ORDINARY DIFFERENTIAL EQUATIONS

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January 12, 2004

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Chapter 1

FIRST ORDER DIFFERENTIAL EQUATIONS

1.1 Introduction

Many of the laws of science and engineering are most readily expressed by describing how some property of interest (position, temperature, population, concentration, etc.) changes over time. This is usually expressed by describing how the rate of change of the quantity is related to the quantity at a particular time. In the language of mathematics, these laws are described by differential equations. An **ordinary differential equation** is an equation relating an unknown function $y(t)$ and some of the derivatives of $y(t)$, and it may also involve the independent variable t, which in many applied problems will represent time. A **partial differential equation** is an equation relating an unknown function $u(\mathbf{t})$ (where the variable $\mathbf{t} = (t_1, \ldots, t_n)$), some of the partial derivatives of u with respect to the variables t_1, ..., t_n, and possibly the variables themselves. In contrast to algebraic equations, where the given and unknown objects are numbers, differential equations belong to the much wider class of **functional equations** in which the given and unknown objects are functions (scalar functions or vector functions).

Example 1.1.1. Each of the following are differential equations:

1. $y' = y - t$

2. $4y'' - 4y' + y = 0$

3. $y'' = yy'$

4. $my'' = f(t)$

5. $\dfrac{\partial^2 u}{\partial t_1^2} + \dfrac{\partial^2 u}{\partial t_2^2} = 0$

6. $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$

The first equation involves the unknown function y, the dependent variable t and the derivative y'. The second, third, and fourth equations involve the unknown function y and the first two derivatives y' and y'', although the first derivative is not explicitly mentioned in the fourth equation. The last two equations are partial differential equations, specifically Laplace's equation and the heat equation, which typically occur in scientific and engineering problems.

In this text we will almost exclusively use the prime notation, that is, y', y'', etc. to denote derivatives. In other sources you may find the Leibnitz notation $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, etc. in use. The objects of study in this text are ordinary differential equations, rather than partial differential equations. Thus, when we use the term differential equation without a qualifying adjective, you should assume that we mean *ordinary* differential equation.

The **order** of a differential equation is the highest order derivative which appears in the equation. Thus, the first equation above has order 1, while the others have order 2. In this course, we shall be primarily concerned with ordinary differential equations (and systems of ordinary differential equations) of order 1 and 2. The **standard form** for an ordinary differential equation is to solve for the highest order derivative as a function of the unknown function y, its lower order derivatives, and the dependent variable t. Thus, a first order ordinary differential equation is given in standard form as

$$y' = F(t, y) \tag{1}$$

while a second order ordinary differential equation in standard form is written

$$y'' = F(t, y, y'). \tag{2}$$

In the previous example, the first and third equations are already in standard form, while the second and fourth equations can be put in standard form by solving for y'':

$$y'' = y' - \frac{1}{4}y$$
$$y'' = \frac{1}{m}f(t).$$

Remark 1.1.2. In applications, differential equations will arise in many forms. The standard form is simply a convenient way to be able to talk about various hypotheses to put on an equation to insure a particular conclusion, such as *existence and uniqueness of solutions* (see Section 1.5), and to classify various types of equations (as we do in the next two sections, for example) so that you will know which algorithm to apply to arrive at a solution.

Remark 1.1.3. We will see that differential equations generally have infinitely many solutions so to specify which solution we are interested in we usually specify an initial value $y(t_0)$ for a first order equation and an initial value $y(t_0)$ and an initial derivative $y'(t_0)$ in the case of a second order equation. When the differential equation and initial values are specified, then one obtains what is known as an **initial value problem**. Thus a first order initial value problem in standard form is

$$y' = F(t, y); \quad y(t_0) = y_0 \tag{3}$$

while a second order equation in standard form is written

$$y'' = F(t, y, y'); \quad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{4}$$

For an algebraic equation, such as $2x^2 + 5x - 3 = 0$, a solution is a particular number which, when substituted into both the left and right hand sides of the equation, gives the same value. Thus, $x = \frac{1}{2}$ is a solution to this equation since

$$2 \cdot \left(\frac{1}{2}\right)^2 + 5 \cdot \left(\frac{1}{2}\right) - 3 = 0$$

while $x = -1$ is *not* a solution since

$$2 \cdot (-1)^2 + 5 \cdot (-1) - 3 = -6 \neq 0.$$

A **solution** of an ordinary differential equation is a function $y(t)$ defined on some specific interval $I = (a, b) \subseteq \mathbb{R}$ such that substituting $y(t)$ for y and substituting $y'(t)$ for y', $y''(t)$ for y'', etc. in the equation gives a **functional identity**. That is, an identity which is satisfied for *all* $t \in I$. For example, if the first order equation is given in standard form as $y' = F(t, y)$, then $y(t)$ defined on $I = (a, b)$ is a solution on I if

$$y'(t) = F(t, y(t)) \quad \text{for all } t \in I,$$

while $y(t)$ is a solution of a second order equation $y'' = F(t, y, y')$ on the interval I if

$$y''(t) = F(t, y(t), y'(t)) \quad \text{for all } t \in I.$$

Example 1.1.4. 1. The function $y_1(t) = 3e^{-2t}$, defined on $(-\infty, \infty)$, is a solution of the differential equation $y' + 2y = 0$ since

$$y_1'(t) + 2y_1(t) = (-2) \cdot 3e^{-2t} + 2 \cdot 3e^{-2t} = 0$$

for *all* $t \in (-\infty, \infty)$, while the function $y_2(t) = 2e^{-3t}$, also defined on $(-\infty, \infty)$, is *not* a solution since

$$y_2'(t) + 2y_2(t) = (-3) \cdot 2e^{-3t} + 2 \cdot 2e^{-3t} = -2e^{-3t} \neq 0.$$

More generally, if c is *any* real number, then the function $y_c(t) = ce^{-2t}$ is a solution to $y' + 2y = 0$ since

$$y_c'(t) + 2y_c(t) = (-2) \cdot ce^{-2t} + 2 \cdot ce^{-2t} = 0$$

for all $t \in (-\infty, \infty)$.

2. The function $y_1(t) = t + 1$ is a solution of the differential equation

$$y' = y - t \qquad (\dagger)$$

on the interval $I = (-\infty, \infty)$ since

$$y_1'(t) = 1 = (t + 1) - t = y_1(t) - t$$

for *all* $t \in (-\infty, \infty)$. The function $y_2(t) = t + 1 - 7e^t$ is also a solution on the same interval since

$$y_2'(t) = 1 - 7e^t = t + 1 - 7e^t - t = y_2(t) - t$$

for all $t \in (-\infty.\infty)$. Note that $y_3(t) = y_1(t) - y_2(t) = 7e^t$ is *not* a solution of $(\dagger)$ since

$$y_3'(t) = 7e^t = y_3(t) \neq y_3(t) - t.$$

There are, in fact, many more solutions to $y' = y - t$. We shall see later that all of the solutions are of the form $y_c(t) = t + 1 + ce^t$ where $c \in \mathbb{R}$ is a constant. Note that y_1 is obtained by taking $c = 0$ and y_2 is obtained by taking $c = 7$. We leave it as an exercise to check that $y_c(t)$ is in fact a solution to $(\dagger)$.

3. The function $y(t) = \tan t$ for $t \in I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a solution of the differential equation $y' = 1 + y^2$ since

$$y'(t) = \frac{d}{dt} \tan t = \sec^2 t = 1 + \tan^2 t = 1 + y(t)^2$$

for all $t \in I$. Note that $z(t) = 2y(t) = 2\tan t$ is not a solution of the same equation since

$$z'(t) = 2\sec^2 t = 2(1 + \tan^2 t) \neq 1 + 4\tan^2 t = 1 + z(t)^2.$$

Note that in this example, the interval on which $y(t)$ is defined, namely $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, is not apparent from looking at the equation $y' = 1 + y^2$. This phenomenon will be explored further in Section 1.5.

4. Consider the differential equation

$$y'' + 16y = 0. \tag{‡}$$

Let $y_1(t) = \cos 4t$. Then

$$y_1''(t) = \frac{d}{dt}(y_1'(t)) = \frac{d}{dt}(-4\sin 4t) = -16\cos 4t = -16y_1(t)$$

so that $y_1(t)$ is a solution of (‡). We leave it as an exercise to check that $y_2(t) = \sin 4t$ and $y_3(t) = 2y_1(t) - y_2(t) = 2\cos 4t - \sin 4t$ are also solutions to (‡). More generally, you should check (as usual by direct substitution) that $y(t) = c_1 \cos 4t + c_2 \sin 4t$ is a solution to (‡) for *any* choice of real numbers c_1 and c_2.

Examples of Differential Equations

We will conclude this introductory section by describing a few examples of situations where differential equations arise in the description of natural phenomena. The goal will be to describe the differential equations or initial value problems which arise, however, we will postpone the *solution* of all but one of the resulting differential equations until later in the chapter when some techniques have been developed. Prior to the examples, we remind you of various useful interpretations of the terms *derivative* and *proportion*, both of which are pervasive in the formulation of mathematical models of natural phenomena.

Remark 1.1.5 (Derivative). In calculus you spent a good deal of time studying what the derivative of a function $y(t)$ is. That is, its "definition" and various interpretations of the derivative, together with rules for calculating the derivative for specific functions. *All* of these are important in understanding and working with differential equations, and not just the rules for calculating derivatives, which may be the part you remember best. The following is a summary of some of the interpretations of derivatives which you will find useful. The function y is defined on an interval $I = (a, b)$ and $t_0 \in I$.

- **Definition:** The derivative of y at t_0 is

$$y'(t_0) = \lim_{t \to t_0} \frac{y(t) - y(t_0)}{t - t_0} \tag{$*$}$$

provided the limit exists. This is the definition you learned in calculus.

- **Rate of Change:** The derivative of y at t_0, $y'(t_0)$, is the instantaneous rate of change of y at t_0. This is the fundamental interpretation of derivative which appears in setting up mathematical models of many natural phenomena. The relationship to the definition of derivative is that the fraction $\dfrac{y(t) - y(t_0)}{t - t_0}$ represents the rate of change of $y(t)$ between times t and t_0 so that the limit $(*)$ is interpreted as the *instantaneous* rate of change of y at t_0.

- **Slope of the tangent line:** The derivative of y at t_0, $y'(t_0)$, is the slope of the tangent line to the graph of the function $y(t)$ at the point $(t_0, y(t_0))$. The relationship to the definition of derivative is that the fraction $\dfrac{y(t) - y(t_0)}{t - t_0}$ is the slope of the secant line joining the two points $(t_0, y(t_0))$ and $(t, y(t))$, so that the limit $(*)$ is interpreted as the slope of the line which best approximates the graph of $y(t)$ at the point $(t_0, y(t_0))$, that is, the slope of the tangent line.

- **Differentiation formulas:** What you may remember best from calculus are the formulas for calculating derivatives for various functions. You will certainly need these in studying differential equations, but the other properties (interpretations) of the derivative are equally necessary. For your convenience, a short table (Table 1.1) of commonly used derivatives (and integrals) is included. For a more extensive table consult your calculus book.

Remark 1.1.6 (Proportion). A commonly used principle in setting up a mathematical model is that of *proportionality*. A function f said to be **proportional** (or **directly proportional** to a function g if $f = kg$ for some constant k. Recall that this means that $f(t) = kg(t)$ for all t in the domain of f. For example, the area of a circle is proportional to the square of the radius (since $A = \pi r^2$), the circumference of a circle is proportional to the radius (since $C = 2\pi r$), the volume of a sphere is proportional to the cube of the radius (since $V = \frac{4}{3}\pi r^3$), and the surface area of a sphere is proportional to the square of the radius (since $S = 4\pi r^2$). Some other variants of proportionality which you are likely to encounter in setting up mathematical models involving differential equations include: f is **inversely proportional** to g if $f(t) = k\dfrac{1}{g(t)}$ where k is a constant; f if proportional to the square of g if $f(t) = k(g(t))^2$ where k is a constant; f is proportional to the square root of g if $f(t) = k\sqrt{g(t)}$ where k is a constant; etc. A simple example you may have seen is the ideal gas law $PV = kT$, which relates the pressure P, volume V, and temperature T of an ideal gas (k is a constant). This equation can be read as several different proportionalities: if P is constant, then V is directly proportional to T; if V is constant, then P is directly proportional to T; if T is constant, then P and V are inversely proportional.

Table 1.1: Table of Derivatives and Integrals

$f(t)$	$f'(t)$	$\int f(t)\,dt$
k	0	$kt + c$
t^n	nt^{n-1}	$\dfrac{t^{n+1}}{n+1} + c$ if $n \neq -1$
$1/t$		$\ln\lvert t\rvert + c$
e^{kt}	ke^{kt}	$\dfrac{e^{kt}}{k} + c$
$\ln\lvert t\rvert$	$1/t$	
$\sin t$	$\cos t$	$-\cos t + c$
$\cos t$	$-\sin t$	$\sin t + c$

One of the main purposes of differential equations in applications is to serve as a tool for the study of change in the physical world. In this context the variable t denotes time and $y(t)$ denotes the state of a physical system at time t. It is a fact of life that humans are not very good in describing "what is", but much better in recognizing how and why things change. A reflection of this metaphysical principle is the fact that many of the laws of physics are expressed in the mathematical language of differential equations, which is another way of saying that one has a formula expressing the way a quantity y changes, rather than giving an explicit description of y. As a first illustration of this basic insight into human nature, we go back to the seventeenth century when the Italian scientist Galileo Galilei (1564 – 1642) dropped stones from the leaning tower of his home town of Pisa. The problem he attempted to solve was to determine the height $y(t)$ at all times t of a stone dropped at time $t_0 = 0$ from height $y(0) = y_0$. After hundreds of experiments which consisted of measurements of time and height when stones were dropped from the tower, he and his co-workers eventually found experimentally that $y(t) = -16t^2 + y_0$. The most important aspect of Galileo's work in describing falling bodies (he also described sliding bodies, planetary orbits, and found two of the laws of motion, among others) was not so much in the derivation of the explicit formulas, but in his tremendous success in popularizing the general idea that physical phenomena could be expressed in mathematical terms. The efforts of Galileo and contemporaries like Johannes Kepler (1571 – 1630), who succeeded after thousands of years of purely observational astronomy to formulate three simple laws governing planetary motion, paved the way for the creation of *calculus* by Gottfried Leibniz (1646 – 1716) and Isaac Newton (1642 – 1727). With calculus available, the derivation of the formula $y(t) = -16t^2 + y_0$ can be done at the desk and with only one experiment performed instead of spending years at a tower dropping stones. It is the first and simplest differential equation of all; it's derivation proceeds as follows.

Example 1.1.7 (Falling Bodies). If $y(t)$ denotes the *position* (position is measured as height above the ground) of a falling body at time t, then its derivative $y'(t)$ denotes the rate of change of position at time t (see Remark 1.1.5). In other words, $y'(t)$ is the falling body's *velocity* at time t. Similarly, since $y''(t)$ denotes the rate of change of velocity, the second derivative $y''(t)$ denotes the falling body's *acceleration* at time t. If the bodies considered are such that air resistance plays only a minor role, then we observe that they hit the ground at the same time if they are dropped at the same time from the same height. Thus, it is not unreasonable to consider the hypothesis that all of these falling bodies experience the same acceleration and the simplest acceleration to postulate is that of constant acceleration. Thus, we arrive at the second order differential equation

$$y''(t) = -g \qquad (*)$$

as our proposed mathematical model for describing the height $y(t)$ of the stone, where we choose the negative sign to indicate that the motion is downwards and not upwards, and we assume that g does not depend on time. Equation $(*)$ is our first differential equation describing how a state (namely $y(t)$) of a physical system (namely, the height of the stone) changes with time. Moreover, the use of calculus makes the solution of $(*)$ straightforward. Indeed, we see immediately (by integrating Equation $(*)$) that

$$y'(t) = -gt + v_0$$

for some constant v_0. Since $y'(0) = v_0$, the constant v_0 denotes the initial velocity of the body. Integrating once more we obtain

$$y(t) = -\frac{g}{2}t^2 + v_0 t + y_0$$

for some constant y_0. Clearly, if we just drop the body, then the initial velocity $v_0 = 0$, and since $y(0) = y_0$, the constant y_0 denotes the initial position (initial height) of the body.

Before we can test our hypothesis against real world data, we have to find the constant g (observe that v_0 and y_0 are known initial data). To do this we go to a window on the second or third floor of a building, measure the height y_0 of the window (in feet), *drop* a stone (i.e., $v_0 = 0$), and measure the time t_h (in seconds) it takes for the stone to hit the ground. Then we go back to our desk and assuming that our hypothesis was true, we conclude that $0 = y(t_h) = -\frac{g}{2}t_h^2 + y_0$ or

$$g = \frac{2y_0}{t_h^2}.$$

If your watch does not need a new battery and if your measurement of the height of the window was not too bad, then the numerical result will be $g = 32$ ft/sec^2 and we come up with the following statement. If a falling body is such that air resistance plays only a minor role, then the assumption $y''(t) = -g$ leads to the conclusion that the position of such a falling body at time t is given by the formula

$$y(t) = -16t^2 + v_0 t + y_0, \tag{$**$}$$

where v_0 denotes the initial velocity and y_0 the initial height of the body. Now you can leave your desk and test this formula against real world measurements. If the computed results match with the observed ones, then you are in luck and you feel more confident in your original hypothesis. If not, you know that your original hypothesis was wrong and you have to go back to the drawing board. Since Galileo came up with the same formula as we did based on his thousands of experiments, we know that we are lucky this time and can stand in awe before our result.

There are many ways humans can describe the motion of a falling body. We can use drawings, music, language, poetry, or $y(t) = -16t^2 + v_0 t + y_0$. The latter may appear to be the most complicated of the methods, but it is vastly superior to others because it is *predictive* (that is, it predicts where you will find the stone at some future time after it is dropped), and therefore it contains within itself the means for justifying its validity. Simply compare actual and predicted positions. Moreover Equation ($**$) is the first step toward what is known as Newton's *second law of motion.* If we assume that a body moves in only one dimension (measured by y) and that the mass m remains constant, the second law can be expressed as

$$y''(t) = \frac{F(t)}{m},$$

where $F(t)$ is the force required to accelerate the body, and $y''(t)$ is the acceleration. Despite its simplicity, this second order differential equation

force equals mass times acceleration

is a cornerstone for treating problems in the physical sciences. □

Example 1.1.8 (Population Growth). Let $P(t)$ denote the population of a given species (e.g., bacteria, rabbits, humans, etc.) at time t in an isolated environment, which simply means that there is no immigration or emigration of the species so that the only changes in population consist of birth and death. If we let b denote the birth rate, that is, the number of births per unit population per unit time, and if we let d denote the death rate, then the change in population between times t_0 and t is given by

$$P(t) - P(t_0) \approx bP(t_0)(t - t_0) - dP(t_0)(t - t_0).$$

Note that each of the two terms on the right hand side is of the form

$$\text{Rate } (= b \text{ or } d) \times \text{Population } (= P(t_0)) \times \text{Time } (= t - t_0).$$

The approximation symbol $\approx$ is used since the birth rate and death rates are not assumed to be constant; in fact, they may very well depend on both time t and the current population P. If we divide by $t - t_0$ and let $t \to t_0$ we find that population growth in an isolated community is governed by the differential equation

$$P'(t) = k(t, P)P(t) \tag{5}$$

where $k(t, P) = b(t, P) - d(t, P)$ is the difference between the instantaneous birth rates and death rates. If we assume that $k(t, P)$ is a constant k, then the differential equation of population growth is $P' = kP$. This model of population growth is known as the *Malthusian model* after the English economist Thomas Robert Malthus (1766 - 1834). An inspection of Table 1.1 shows that $P(t) = e^{kt}$ is one solution of this equation since, in this case $P'(t) = ke^{kt} = kP(t)$ for all $t \in \mathbb{R}$. Similarly, $P(t) = ce^{kt}$ is also a solution for *any* $c \in \mathbb{R}$. Since $P(0) = c$ the meaning of the constant c is that it is the population at time 0. We shall see in the next section that the solutions $P(t) = ce^{kt}$ are *all* of the solutions of the population equation $P' = kP$ when k is a constant (known as the *growth rate* of the population).

Continuing with the assumption that the growth rate k is constant, if $k > 0$ then $P(t) = P_0e^{kt}$ (where $P_0 = P(0)$) and the population grows without bound (recall that $\lim_{t\to\infty} e^{kt} = \infty$ if $k > 0$). This is unrealistic since populations of any species will be limited by space and food, so we should try to modify it to obtain a differential equation whose solutions are more in line with observed population data. If we can do so, then we might also have some confidence that any predictions made for the future population will also have some validity. One *possible* model that one can devise is to assume that the environment will support a maximum population, call it M, and then we can assume that the growth rate is proportional to how close the population is to the maximum supportable population M. This can be expressed as an equation by

$$k(t, P) = c(M - P),$$

where c is a proportionality constant. With this assumption, Equation (5) becomes

$$P' = c(M - P)P. \tag{6}$$

This model of population growth was first introduced by the Belgian mathematician Pierre Verhulst (1804 – 1849). Is this a better model for population growth than the simple constant growth model $P' = kP$? At this point we can't answer this question since, unlike the constant growth model $P' = kP$, it is not so easy to guess what solutions to Equation (6) look like. As it turns out, Equation (6) is one of the types of equations which we can solve explicitly. We shall do so in Section 1.2. □

Example 1.1.9. Consider a tank which contains 2000 gallons of water in which 10 lbs of salt are dissolved. Suppose that a water-salt mixture containing 0.1 lb/gal enters the tank at a rate of 2 gal/min, and assume that the well-stirred mixture flows from the tank at the same rate of 2 gal/min. Find an initial value problem to describe the amount $y(t)$ of salt (expressed in pounds) which is present in the tank at all times t measured in minutes after the initial time ($t = 0$) when 10 lbs are present.

► **Solution.** This is another example of where it is easier to describe how $y(t)$ changes, that is $y'(t)$, than it is to directly describe $y(t)$. Since the description of $y'(t)$ will also include $y(t)$, a differential equation will result. Start by noticing that at time t_0, $y(t_0)$ lbs of salt are present and at a later time t, the amount of salt in the tank is given by

$$y(t) = y(t_0) + A(t_0, t) - S(t_0, t)$$

where $A(t_0, t)$ is the amount of salt added between times t_0 and t and $S(t_0, t)$ is the amount removed between times t_0 and t. To compute $A(t_0, t)$ note that

$$A(t_0, t) = (\text{Number of lbs/gal})\cdot(\text{Number of gal/min})\cdot(\text{Number of minutes from } t_0 \text{ to } t)$$

so that

$$A(t_0, t) = (0.1) \cdot (2) \cdot (t - t_0).$$

By exactly the same reasoning,

$$S(t_0, t) = (\text{Number of lbs/gal})\cdot(\text{Number of gal/min})\cdot(\text{Number of minutes from } t_0 \text{ to } t).$$

The number of gallons per minute flowing out of the tank is still 2 gal/min. However, the number of pounds per gallon at any given time t will be given by $y(t)/V(t)$, that is divide the total number of pounds of salt in the tank at time t by the current total volume $V(t)$ of solution in the tank. In our case, $V(t)$ is always 2000 gal (the flow in and the flow out balance), but $y(t)$ is constantly changing and that is what we ultimately will want to compute. If t is "close" to t_0 then we can assume that $y(t) \approx y(t_0)$ so that

$$S(t_0, t) \approx \left(\frac{y(t_0)}{2000}\text{lbs/gal}\right) \cdot (2\text{gal/min}) \cdot (t - t_0).$$

Combining all of these results gives

$$\begin{aligned} y(t) - y(t_0) &= A(t_0, t) - S(t_0, t) \\ &\approx (0.2)(t - t_0) - 2\frac{y(t_0)}{2000}(t - t_0). \end{aligned}$$

Dividing this by $t - t_0$ and letting $t \to t_0$ gives the equation

$$y'(t_0) = 0.2 - \frac{1}{1000}y(t_0),$$

which we recognize as a differential equation. Note that it is the process of taking the limit as $t \to t_0$ that allows us to return to an equation, rather than dealing only with an approximation. This is a manifestation of what we mean when we indicate that it is frequently easier to describe the way something changes, that is $y'(t)$, rather than "what is," i.e. $y(t)$ itself.

Since t_0 is an arbitrary time, we can write the above equation as a differential equation

$$y' = (0.2) - \frac{1}{1000}y \tag{7}$$

and it becomes an initial value problem by specifying that we want $y(0) = 10$, that is, there are 10 lbs of salt initially present in the tank.

The differential equation obtained is an example of what is known as a *first order linear differential equation*. This is an important class of differential equations which we will study in detail in Section 1.3. At that time we shall return to this example and solve Equation (7). ◀

We will conclude this section by summarizing a slightly more general situation than that covered by the previous numerical example.

Example 1.1.10 (Mixing problem). A tank initially holds V_0 gal of brine(a water-salt mixture) that contains a lb of salt. Another brine solution, containing c lb of salt per gallon, is poured into the tank at a rate of r gal/min. The mixture is stirred to maintain uniformity of concentration of salt at all parts of the tank, and the stirred mixture flows out of the tank at the rate of R gal/min. Let $y(t)$ denote the amount of salt (measured in pounds) in the tank at time t. Find an initial value problem for $y(t)$.

▶ **Solution.** We are searching for an equation which describes the *rate of change of the amount of salt in the tank at time* t, i.e., $y'(t)$. The key observation is that this rate of change is the difference between the rate at which salt is being added to the tank and the rate at which the salt is being removed from the tank. In symbols:

$$y'(t) = \text{Rate in} - \text{Rate out}.$$

The rate that salt is being added is easy to compute. It is rc lb/min (c lb/gal $\times r$ gal/min $= rc$ lb/min). Note that this is the appropriate units for a rate, namely an amount divided by a time. We still need to compute the rate at which salt is leaving the tank. To do this we first need to know the number of gallons $V(t)$ of brine in the tank at time t. But this is just the initial volume plus the amount added up to time t minus the amount removed up to time t. That is, $V(t) = V_0 + rt - Rt = V_0 + (r - R)t$. Since $y(t)$ denotes the amount of salt present in the tank at time t, the concentration of salt at time t is $y(t)/V(t) = y(t)/(V_0 - (r - R)t)$, and the rate at which salt leaves the tank is $R \times y(t)/V(t) = Ry(t)/(V_0 + (r - R)t)$. Thus,

$$\begin{aligned} y'(t) &= \text{Rate in} - \text{Rate out} \\ &= rc - \frac{R}{V_0 + (r - R)t} y(t) \end{aligned}$$

In the standard form of a linear differential equation, the equation for the rate of change of $y(t)$ is

$$\boxed{y'(t) + \frac{R}{V_0 + (r - R)t} y(t) = rc.} \tag{8}$$

This becomes an initial value problem by remembering that $y(0) = a$. As in the previous example, this is a first order linear differential equation, and the solutions will be studied in Section 1.3. ◀

Remark 1.1.11. You should definitely **not** memorize a formula like Equation (8). What you should remember is how it was set up so that you can set up your own problems, even if the circumstances are slightly different from the one given above. As one example of a possible variation, you might encounter a situation in which the volume $V(t)$ varies in a nonlinear manner such as, for example, $V(t) = 5 + 3e^{-2t}$.

Exercises

What is the order of each of the following differential equations?

1. $y^2 y' = t^3$

2. $y'y'' = t^3$

3. $t^2 y' + ty = e^t$

4. $t^2 y'' + ty' + 3y = 0$

5. $3y' + 2y + y'' = t^2$

Determine whether each of the given functions $y_j(t)$ is a solution of the corresponding differential equation.

6. $y' = 2y$: $\quad y_1(t) = 2, \quad y_2(t) = t^2, \quad y_3(t) = 3e^{2t}, \quad y_4(t) = 2e^{3t}.$

7. $y' = 2y - 10$: $\quad y_1(t) = 5, \quad y_2(t) = 0, \quad y_3(t) = 5e^{2t}, \quad y_4(t) = e^{2t} + 5.$

8. $ty' = y$: $\quad y_1(t) = 0, \quad y_2(t) = 3t, \quad y_3(t) = -t, \quad y_4(t) = t^3.$

9. $y'' + 4y = 0$: $\quad y_1(t) = e^{2t}, \quad y_2(t) = \sin 2t, \quad y_3(t) = \cos(2t - 1), \quad y_4(t) = t^2.$

Verify that each of the given functions $y(t)$ is a solution of the given differential equation on the given interval.

10. $y' = 3y + 12 \qquad y(t) = ce^{3t} - 4 \qquad$ for $t \in (-\infty, \infty)$, $c \in \mathbb{R}$

11. $y' = -y + 3t \qquad y(t) = ce^{-t} + 3t - 3 \qquad$ for $t \in (-\infty, \infty)$, $c \in \mathbb{R}$

12. $y' = y^2 - y \qquad y(t) = 1/(1 - ce^t) \qquad$ as long as the denominator is not 0, $c \in \mathbb{R}$

13. $y' = 2ty \qquad y(t) = ce^{t^2} \qquad$ for $t \in (-\infty, \infty)$, $c \in \mathbb{R}$

14. $(t + 1)y' + y = 0 \qquad y(t) = c(t + 1)^{-1} \qquad$ for $t \in (-1, \infty)$, $c \in \mathbb{R}$

Find the general solution of each of the following differential equations by integration. (See the solution of Equation $(*)$ in Example 1.1.7.)

15. $y' = t + 3$

▶ **Solution.** $y(t) = \int y'(t)\, dt = \int (t + 3)\, dt = \frac{t^2}{2} + 3t + c$ ◀

16. $y' = e^{2t} - 1$

17. $y' = te^{-t}$

18. $y' = \dfrac{t+1}{t}$

19. $y'' = 2t + 1$

20. $y'' = 6 \sin 3t$

Find a solution to each of the following initial value problems. See Exercises 10 through 20 for the general solutions of these equations.

21. $y' = 3y + 12$, $y(0) = -2$

▶ **Solution.** The general solution is $y(t) = ce^{3t} - 4$ from Exercise 10. $-2 = y(0) = c - 4 \implies c = 2$, so $y(t) = 2e^{3t} - 4$. ◀

22. $y' = -y + 3t, \quad y(0) = 0$

23. $y' = y^2 - y, \quad y(0) = 1/2$

24. $(t+1)y' + y = 0, \quad y(1) = -9$

25. $y' = e^{2t} - 1, \quad y(0) = 4$

26. $y' = te^{-t}, \quad y(0) = -1$

27. $y'' = 6 \sin 3t, \quad y(0) = 1, \, y'(0) = -2$

28. Radium decomposes at a rate proportional to the amount present. Express this proportionality statement as a differential equation for $R(t)$, the amount of radium present at time t.

29. One kilogram of sugar dissolved in water is being transformed into dextrose at a rate which is proportional to the amount not yet converted. Write a differential equation satisfied by $y(t)$, the amount of sugar present at time t. Make it an initial value problem by giving $y(0)$.

30. Bacteria are placed in a sugar solution at time $t = 0$. Assuming adequate food and space for growth, the bacteria will grow at a rate proportional to the current population of bacteria. Write a differential equation satisfied by the number $P(t)$ of bacteria present at time t.

31. Continuing with the last exercise, assume that the food source for the bacteria is adequate, but that the colony is limited by space to a maximum population M. Write a differential equation for the population $P(t)$ which expresses the assumption that the

growth rate of the bacteria is proportional to the product of the number of bacteria currently present and the difference between M and the current population.

32. Newton's law of cooling states that the rate at which a body cools (or heats up) is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. If a bottle of your favorite beverage is at room temperature (say 70° F) and it is then placed in a tub of ice at time $t = 0$, write an initial value problem which is satisfied by the temperature $T(t)$ of the bottle at time t.

33. On planet P the following experiment is performed. A small rock is dropped from a height of 4 feet and it is observed that it hits the ground in 1 sec. Suppose another stone is dropped from a height of 1000 feet. What will be the height after 5 sec.? How long will it take for the stone to hit the ground.

1.2 Separable Equations

In this section and the next we shall illustrate how to obtain solutions for two particularly important classes of first order differential equations. Both classes of equations are described by means of restrictions on the type of function $F(t, y)$ which appears on the right hand side of a first order ordinary differential equation given in standard form

$$y' = F(t, y). \tag{1}$$

The simplest of the standard types of first-order equations are those with **separable variables**; that is, equations of the form

$$y' = h(t)g(y). \tag{2}$$

Such equations are said to be **separable equations**. Thus, an equation $y' = F(t, y)$ is a separable equation provided that the right hand side $F(t, y)$ can be written a a product of a function of t and a function of y. Most functions of two variables are *not* the product of two one variable functions.

Example 1.2.1. Identify the separable equations from among the following list of differential equations.

1. $y' = t^2y^2$
2. $y' = t^2 + y$

3. $y' = \dfrac{t-y}{t+y}$
4. $y' = y - y^2$
5. $(2t-1)(y^2-1)y' + t - y - 1 + ty = 0$
6. $y' = f(t)$
7. $y' = p(t)y$
8. $y'' = ty$

► **Solution.** Equations 1, 4, 5, 6, and 7 are separable. For example, in Equation 4, $h(t) = 1$ and $g(y) = y - y^2$, while, in Example 6, $h(t) = f(t)$ and $g(y) = 1$. To see that Equation 5 is separable, we bring all terms not containing y' to the other side of the equation; i.e.,

$$(2t-1)(y^2-1)y' = -t + y + 1 - ty = -t(1+y) + 1 + y = (1+y)(1-t).$$

Solving this equation for y' gives

$$y' = \frac{(1-t)}{(2t-1)} \cdot \frac{(1+y)}{(y^2-1)},$$

which is clearly separable. Equations 2 and 3 are not separable since neither right hand side can be written as product of a function of t and a function of y. Equation 8 is not a separable equation, even though the right hand side is $ty = h(t)g(y)$, since it is a *second* order equation and our definition of separable applies only to first order equations. ◄

Equation 6 in the above example, namely $y' = f(t)$ is particularly simple to solve. This is precisely the differential equation that you spent half of your calculus course understanding, both what it means and how to solve it for a number of common functions $f(t)$. Specifically, what we are looking for in this case is an **antiderivative** of the function $f(t)$, that is, a function $y(t)$ such that $y'(t) = f(t)$. Recall from calculus that if $f(t)$ is a continuous function on an interval $I = (a, b)$, then the Fundamental Theorem of Calculus guarantees that there is an antiderivative of $f(t)$ on I. Let $F(t)$ be *any* antiderivative of $f(t)$ on I. Then, if $y(t)$ is any solution to $y' = f(t)$, it follows that $y'(t) = f(t) = F'(t)$ for all $t \in I$. Since two functions which have the same derivatives on an interval I differ by a constant c, we see that the general solution to $y' = f(t)$ is

$$\boxed{y(t) = F(t) + c.} \tag{3}$$

There are a couple of important comments to make concerning Equation (3).

1. The antiderivative of f exists on *any* interval I on which f is continuous. This is the main point of the Fundamental Theorem of Calculus. Hence the equation $y' = f(t)$ has a solution on any interval I on which the function f is continuous.

2. The constant c in Equation 3 can be determined by specifying $y(t_0)$ for some $t_0 \in I$. For example, the solution to $y' = 6t^2$, $y(-1) = 3$ is $y(t) = 2t^3 + c$ where $3 = y(-1) = 2(-1)^3 + c$ so $c = 5$ and $y(t) = 2t^3 + 5$.

3. The indefinite integral notation is frequently used for antiderivatives. Thus the equation
$$y(t) = \int f(t)\,dt$$
just means that $y(t)$ is an antiderivative of $f(t)$. In this notation the constant c in Equation 3 is implicit, although in some instances we may write out the constant c explicitly for emphasis.

4. The formula $y(t) = \int f(t)\,dt$ is valid even if the integral cannot be computed in terms of elementary functions. In such a case, you simply leave your answer expressed as an integral, and if numerical results are needed, you can use numerical integration. Thus, the only way to describe the solution to the equation $y' = e^{t^2}$ is to express the answer as
$$y(t) = \int e^{t^2}\,dt.$$
The indefinite integral notation we have used here has the constant of integration implicitly included. One can be more precise by using a definite integral notation, as in the Fundamental Theorem of Calculus. With this notation,
$$y(t) = \int_{t_0}^{t} e^{u^2}\,du + c, \quad y(t_0) = c.$$

We now extend the solution of $y' = f(t)$ by antiderivatives to the case of a general separable equation $y' = h(t)g(y)$, and we provide an algorithm for solving this equation.

Suppose $y(t)$ is a solution on an interval I of Equation (2), which we write in the form
$$\frac{1}{g(y)}y' = h(t),$$
and let $Q(y)$ be an antiderivative of $\dfrac{1}{g(y)}$ as a function of y, i.e., $Q'(y) = \dfrac{dQ}{dy} = \dfrac{1}{g(y)}$ and let H be an antiderivative of h. It follows from the chain rule that
$$\frac{d}{dt}Q(y(t)) = Q'(y(t))y'(t) = \frac{1}{g(y(t))}y'(t) = h(t) = H'(t).$$

This equation can be written as

$$\frac{d}{dt}\left(Q(y(t)) - H(t)\right) = 0.$$

Since a function with derivative equal to zero on an interval is a constant, it follows that the solution $y(t)$ is implicitly given by the formula

$$Q(y(t)) = H(t) + c. \tag{4}$$

Conversely, assume that $y(t)$ is any function which satisfies the implicit equation (4). Differentiation of both sides of Equation (4) gives, (again by the chain rule),

$$h(t) = H'(t) = \frac{d}{dt}(Q(y(t))) = Q'(y(t))y'(t) = \frac{1}{g(y(t))}y'(t).$$

Hence $y(t)$ is a solution of Equation (2).

Note that the analysis in the previous two paragraphs is valid as long as $h(t)$ and $q(y) = \dfrac{1}{g(y)}$ have antiderivatives. From the Fundamental Theorem of Calculus, we know that a sufficient condition for this to occur is that h and q are *continuous* functions, and q will be continuous as long as g is continuous and $g(y) \neq 0$. We can thus summarize our results in the following theorem.

Theorem 1.2.2. *Let g be continuous on the interval $J = \{y : c \le y \le d\}$ and let h be continuous on the interval $I = \{t : a \le t \le b\}$. Let H be an antiderivative of h on I, and let Q be an antiderivative of $\dfrac{1}{g}$ on an interval $J' \subseteq J$ for which $y_0 \in J'$ and $g(y_0) \neq 0$. Then $y(t)$ is a solution to the initial value problem*

$$y' = h(t)g(y); \quad y(t_0) = y_0 \tag{5}$$

if and only if $y(t)$ is a solution of the implicit equation

$$\boxed{Q(y(t)) = H(t) + c,} \tag{6}$$

where the constant c is chosen so that the initial condition is satisfied. Moreover, if y_0 is a point for which $g(y_0) = 0$, then the constant function $y(t) \equiv y_0$ is a solution of Equation (5)*.*

Proof. The only point not covered in the paragraphs preceding the theorem is the case where $g(y_0) = 0$. But if $g(y_0) = 0$ and $y(t) = y_0$ for all t, then

$$y'(t) = 0 = h(t)g(y_0) = h(t)g(y(t))$$

for all t. Hence the constant function $y(t) = y_0$ is a solution of Equation (5). □

We summarize these observations in the following separable equation algorithm.

Algorithm 1.2.3 (Separable Equation). To solve a separable differential equation, perform the following operations.

1. First put the equation in the form

   $$\text{(I)} \qquad y' = \frac{dy}{dt} = h(t)g(y),$$

   if it is not already in that form.

2. Then we separate variables in a form convenient for integration, i.e. we formally write

   $$\text{(II)} \qquad \frac{1}{g(y)}\, dy = h(t)\ dt.$$

   Equation (II) is known as the "differential" form of Equation (I).

3. Next we integrate both sides of Equation (II) (the left side with respect to y and the right side with respect to t) and introduce a constant c, due to the fact that antiderivatives coincide up to a constant. This yields

   $$\text{(III)} \qquad \int \frac{1}{g(y)}\, dy = \int h(t)\, dt + c.$$

4. Now evaluate the antiderivatives and solve the resulting implicit equation for y as a function of t, if you can (this won't always be possible).

5. Additionally, the numbers y_0 with $g(y_0) = 0$ will give constant solutions $y(t) \equiv y_0$ that will not be seen from the general algorithm. □

Example 1.2.4. Find the solutions of the differential equation $y' = \dfrac{t}{y}$.

► **Solution.** We first rewrite the equation in the form

$$\frac{dy}{dt} = \frac{t}{y} \tag{I}$$

and then in differential form as

$$y\,dy = t\,dt. \tag{II}$$

Integration of both sides of Equation (II) gives

$$\int y\,dy = \int t\,dt + c \tag{III}$$

or

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + c.$$

Multiplying by 2 we get $y^2 = t^2 + c$, where we write c instead of $2c$ since twice an arbitrary constant c is still an arbitrary constant. Thus, if a function $y(t)$ satisfies the differential equation $yy' = t$, then

$$y(t) = \pm\sqrt{t^2 + c} \tag{$*$}$$

for some constant $c \in \mathbb{R}$. On the other hand, since all functions of the form $(*)$ solve $yy' = t$, it follows that the solutions are given by $(*)$. Figure 1.1 shows several of the curves $y^2 = t^2 + c$ which implicitly define the solutions of $yy' = t$. Note that each of the curves in the upper half plane is the graph of $y(t) = \sqrt{t^2 + c}$ for some c, while each curve in the lower half plane is the graph of $y(t) = -\sqrt{t^2 + c}$. None of the solutions are defined on the t-axis, i.e., when $y = 0$. Notice that each of the solutions is an arm of the hyperbola $y^2 - t^2 = c$. ◄

Example 1.2.5. Solve the differential equation $y' = ky$ where $k \in \mathbb{R}$ is a constant.

► **Solution.** First note that the constant function $y = 0$ is one solution. When $y \neq 0$ we rewrite the equation in the form $\frac{y'}{y} = k$, which in differential form becomes

$$\frac{1}{y}\,dy = k\,dt.$$

Integrating both sides of this equation (the left side with respect to y and right side with respect to t) gives

$$\ln|y| = kt + c. \tag{$\dagger$}$$

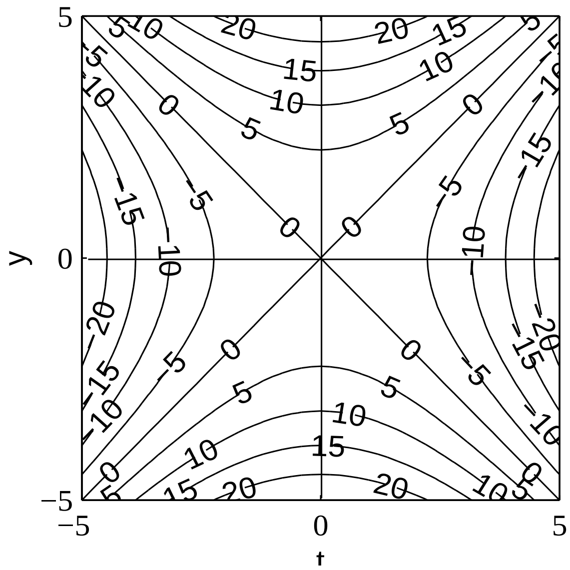


Figure 1.1: The solutions of $yy' = t$ are the level curves of $y^2 = t^2 + c$. The constant c is labeled on each curve.

Applying the exponential function to both sides of (†), and recalling that $e^{\ln x} = x$ for all $x > 0$, we see that

$$|y| = e^{\ln|y|} = e^{kt+c} = e^c e^{kt},$$

so that

$$y = \pm e^c e^{kt}. \tag{‡}$$

Since c is an arbitrary constant, e^c is an arbitrary *positive* constant, so $\pm e^c$ is an arbitrary nonzero constant, which (as usual) we will continue to denote by c. Thus we can rewrite Equation (‡) as

$$y = ce^{kt}. \tag{7}$$

Letting $c = 0$ will give the solution $y = 0$ of $y' = ky$. Thus, as c varies over $\mathbb{R}$, Equation (7) describes *all* solutions of the differential equation $y' = ky$. Note that $c = y(0)$ is the initial value of y. Hence, the solution of the initial value problem $y' = ky$, $y(0) = y_0$ is

$$\boxed{y(t) = y_0 e^{kt}.} \tag{8}$$

Figure 1.2 illustrates a few solution curves for this equation. ◀

A concrete example of the equation $y' = ky$ is given by radioactive decay.

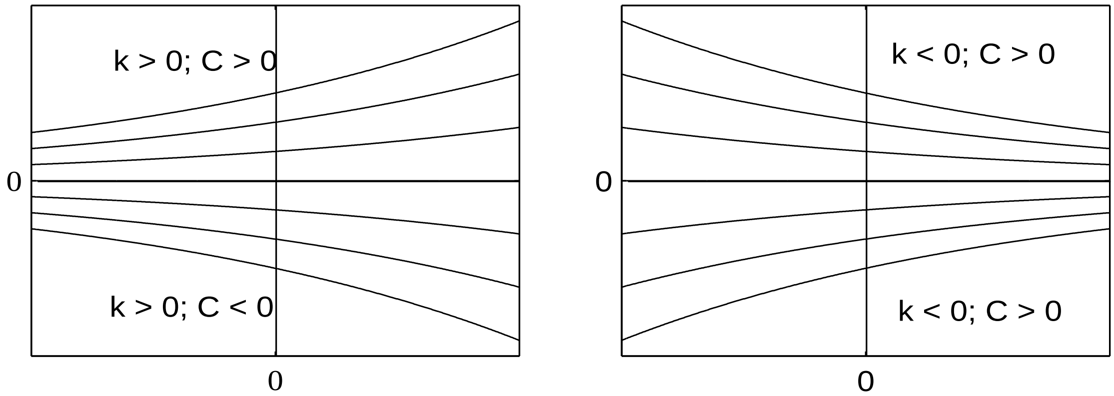


Figure 1.2: Some solutions of $y' = ky$ for various $y(0) = c$. The left picture is for $k > 0$, the right for $k < 0$.

Example 1.2.6 (Radioactive Decay). Suppose that a quantity of a radioactive substance originally weighing y_0 grams decomposes at a rate proportional to the amount present and that half the quantity is left after a years (a is the so-called **half-life** of the substance). Find the amount $y(t)$ of the substance remaining after t years. In particular, find the number of years it takes such that $1/n$-th of the original quantity is left.

▶ **Solution.** Since the rate of change $y'(t)$ is proportional to the amount $y(t)$ present, we are led to the initial value problem

$$y' = -ky\,,\; y(0) = y_0,$$

with solution $y(t) = y_0 e^{-kt}$, where k is a positive constant yet to be determined (the minus sign reflects the observation that $y(t)$ is decreasing as t is increasing). Since $y(a) = \dfrac{y_0}{2} = e^{-ka}$, it follows that $k = \dfrac{\ln 2}{a}$. Thus,

$$y(t) = y_0 2^{-\frac{t}{a}}.$$

This yields easily $t = \frac{a \ln n}{\ln 2}$ as the answer to the last question by solving $y_0 2^{-\frac{t}{a}} = \frac{y_0}{n}$ for t. ◀

Example 1.2.7. Solve the differential equation $(2t-1)(y^2-1)y' + t - y - 1 + ty = 0$.

▶ **Solution.** To separate the variables in this equation we bring all terms not containing y' to the right hand side of the equation, so that

$$(2t-1)(y^2-1)y' = -t + y + 1 - ty = -t(1+y) + 1 + y = (1+y)(1-t).$$

This variables can now be separated, yielding

$$\frac{y^2-1}{1+y}y' = \frac{1-t}{2t-1}.$$

Before further simplification, observe that the constant function $y(t) = -1$ is a solution of the original problem. If we now consider a solution other than $y(t) = -1$, the equation can be written in differential form (after expanding the right hand side in a partial fraction) as

$$(y-1)\,dy = \left(-\frac{1}{2} + \frac{1}{2}\frac{1}{2t-1}\right)\,dt.$$

Integrating both sides of the equation gives, $\frac{1}{2}y^2 - y = -\frac{t}{2} + \frac{1}{4}\ln|2t-1| + c$. Solving for y (and renaming the constant several times) we obtain the general solution as either $y(t) = -1$ or

$$y(t) = 1 \pm \sqrt{c - t + \frac{1}{2}\ln|2t-1|}.$$

◀

Example 1.2.8. Solve the Verhulst population equation $p' = r(m-p)p$ (Equation (6)) where r and m are positive constants.

▶ **Solution.** Since

$$\frac{1}{(m-p)p} = \frac{1}{m}\left(\frac{1}{p} + \frac{1}{m-p}\right),$$

the equation can be written with separated variables in differential form as

$$\frac{1}{(m-p)p}\,dp = \frac{1}{m}\left(\frac{1}{p} + \frac{1}{m-p}\right)\,dp = r\,dt,$$

and the differential form is integrated to give

$$\frac{1}{m}\left(\ln|p| - \ln|m-p|\right) = rt + c,$$

where c is an arbitrary constant of integration. Multiplying by m and renaming mc as c (to denote an arbitrary constant) we get

$$\ln\left|\frac{p}{m-p}\right| = rmt + c,$$

and applying the exponential function to both sides of the equation gives

$$\left|\frac{p}{m-p}\right| = e^{rmt+c} = e^c e^{rmt},$$

or

$$\frac{p}{m-p} = \pm e^c e^{rmt}.$$

Since c is an arbitrary real constant, it follows that $\pm e^c$ is an arbitrary real nonzero constant, which we will again denote by c. Thus, we see that p satisfies the equation

$$\frac{p}{m-p} = ce^{rmt}.$$

Solving this equation for p, we find that the general solution of the Verhulst population equation (6) is given by

$$p(t) = \frac{cme^{rmt}}{1+ce^{rmt}}. \tag{9}$$

Multiplying the numerator and denominator by e^{-rmt}, we may rewrite Equation (9) in the equivalent form

$$p(t) = \frac{cm}{c+e^{-rmt}}. \tag{10}$$

Some observations concerning this equation:

1. The constant solution $p(t) = 0$ is obtained by setting $c = 0$ in Equation (10), even though $c = 0$ did not occur in our derivation.

2. The constant solution $p(t) = m$ does not occur for *any* choice of c, so this solution is an extra one.

3. Note that

$$\lim_{t\to\infty} p(t) = \frac{cm}{c} = m,$$

independent of $c \neq 0$. What this means is that if we start with a positive population, then over time, the population will approach a maximum (sustainable) population m.

4. Figure 1.2 shows the solution of the Verhulst population equation $y' = y(3-y)$ with initial population $y(0) = 1$. You can see from the graph that $y(t)$ approaches the limiting population 3 as t grows. It appears that $y(t)$ actually equals 3 after some point, but this is not true. It is simply a reflection of the fact that $y(t)$ and 3 are so close together that the lines on a graph cannot distinguish them.

◀

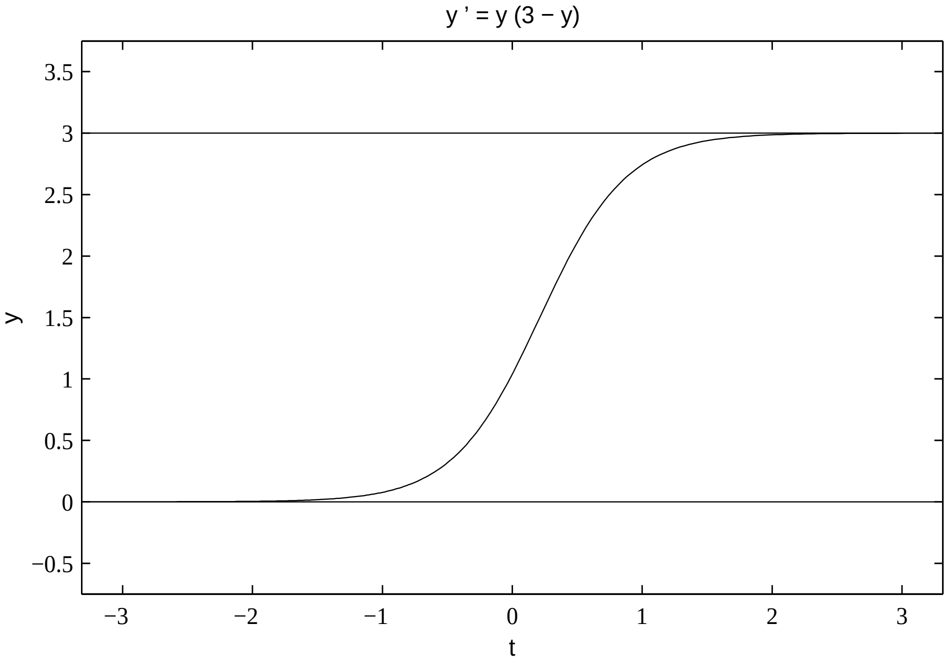


Figure 1.3: Solution of the population problem $y' = y(3-y)$, $y(0) = 1$

Exercises

In each of the following problems determine whether or not the equation is separable. Do **not** solve the equations!!

1. $y' = 2y(5-y)$

2. $t^2y' = 1 - 2ty$

3. $yy' = 1 - y$

4. $\dfrac{y'}{y} = y - t$

5. $ty' = y - 2ty$

6. $(t^2 + 3y^2)y' = -2ty$

7. $y' = ty^2 - y^2 + t - 1$

8. $y' = t^2 + y^2$

9. $e^t y' = y^3 - y$

Find the general solution of each of the following differential equations. If an initial condition is given, find the particular solution which satisfies this initial condition.

10. $yy' = t$, $y(2) = -1$.

▶ **Solution.** The variables are already separated, so integrate both sides of the equation to get $\frac{1}{2}y^2 = \frac{1}{2}t^2 + c$, which we can rewrite as $y^2 - t^2 = k$ where $k \in \mathbb{R}$ is a constant. Since $y(2) = -1$, it follows that $k = (-1)^2 - 2^2 = -3$ so the solution is given implicitly by the equation $y^2 - t^2 = -3$ or we can solve explicitly to get $y = -\sqrt{t^2 - 3}$, where the negative square root is used since $y(2) = -1 < 0$. ◀

11. $(1 - y^2) - tyy' = 0$

▶ **Solution.** It is first necessary to separate the variables by rewriting the equation as $tyy' = (1 - y^2)$. This gives an equation

$$\frac{y}{1 - y^2} y' = \frac{1}{t},$$

or in the language of differentials:

$$\frac{y}{1 - y^2} dy = \frac{1}{t} dt.$$

Integrating both sides of this equation gives

$$-\frac{1}{2} \ln |1 - y^2| = \ln |t| + c.$$

Multiplying by -2, and taking the exponential of both sides gives an equation $|1 - y^2| = \pm kt^{-2}$ where k is a positive constant. By considering an arbitrary constant (which we will call c), this can be written as an implicit equation $t^2(1 - y^2) = c$. ◀

12. $y^3 y' = t$

13. $y^4 y' = t + 2$

14. $y' = ty^2$

15. $y' = t^2 y^2$

16. $y' + (\tan t)y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

17. $y' = t^m y^n$, where m and n are positive integers, $n \neq 1$.

18. $y' = 4y - y^2$

19. $yy' = y^2 + 1$

20. $y' = y^2 + 1$

21. $tyy' + t^2 + 1 = 0$

22. $y + 1 + (y - 1)(1 + t^2)y' = 0$

23. $2yy' = e^t$

24. $(1 - t)y' = y^2$

25. $ty - (t + 2)y' = 0$

Solve the following initial value problems:

26. $\frac{dy}{dt} - y = y^2, \quad y(0) = 0.$

27. $y' = 4ty^2, \quad y(1) = 0$

28. $\frac{dy}{dx} = \frac{xy+2y}{x}, \quad y(1) = e$

29. $y' + 2yt = 0, \quad y(0) = 4$

30. $y' = \frac{\cot y}{t}, \quad y(1) = \frac{\pi}{4}$

31. $\frac{(u^2+1)}{y}\frac{dy}{du} = u, \quad y(0) = 2$

In the following problem you may assume Newton's Law of Heating and cooling. (See Exercise 32 in Section 1.1.)

32. A turkey, which has an initial temperature of 40° (Fahrenheit), is placed into a 350° oven. After one hour the temperature of the turkey is 120°. Use Newton's Law of heating and cooling to find (1) the temperature of the turkey after 2 hours, and (2) how many hours it takes for the temperature of the turkey to reach 250°.

▶ **Solution.** Recall that Newton's Law of heating and cooling states: *The change in the temperature of an object is proportional to the difference between the temperature of the object and the temperature of the surrounding medium.* Thus, if $T(t)$ is the temperature of the object at time t and T_s is the temperature of the surrounding medium then

$$T'(t) = r(T - T_s),$$

for some proportionality constant r. Applying this to the problem at hand, the oven is the surrounding medium and has a constant temperature of 350°. Thus $T_s = 350$ and the differential equation that describes T is

$$T' = r(T - 350).$$

This equation is separable and the solution is

$$T(t) = 350 + ke^{rt},$$

where k is a constant. The initial temperature of the turkey is 40°. Thus, $40 = T(0) = 350 + k$ and this implies $k = -310$. Therefore $T(t) = 350 - 310e^{rt}$. To determine r note that we are given $T(1) = 120$. This implies $120 = T(1) = 350 - 310e^{r}$ and solving for r gives $r = \ln\frac{23}{31} \approx -.298$. To answer question (1), compute $T(2) = 350 - 310e^{2r} \approx 179.35°$. To answer question (2), we want to find t so that $T(t) = 250$, i.e, solve $250 = T(t) = 350 - 310e^{rt}$. Solving this gives $rt = \ln\frac{10}{31}$ so $t \approx 3.79$ hours. ◀

33. A cup of coffee, brewed at 180° (Fahrenheit), is brought into a car with inside temperature 70°. After 3 minutes the coffee cools to 140°. What is the temperature 2 minutes later?

34. The temperature outside a house is 90° and inside it is kept at 65°. A thermometer is brought from the outside reading 90° and after 10 minutes it reads 85°. How long will it take to read 75°? What will the thermometer read after an hour?

35. A cold can of soda is taken out of a refrigerator with a temperature of 40° and left to stand on the countertop where the temperature is 70°. After 2 hours the temperature of the can is 60°. What was the temperature of the can 1 hour after it was removed from the refrigerator?

36. A large cup hot of coffee is bought from a local drive through restaurant and placed in a cup holder in a vehicle. The inside temperature of the vehicle is 70° Fahrenheit. After 5 minutes the driver spills the coffee on himself a receives a severe burn. Doctors determine that to receive a burn of this severity, the temperature of the coffee must have been about 150°. If the temperature of the coffee was 142° 6 minutes after it was sold what was the temperature at which the restaurant served it.

37. A student wishes to have some friends over to watch a football game. She wants to have cold beer ready to drink when her friends arrive at 4 p.m. According to her tastes the temperature of beer can be served when its temperature is 50°. Her experience shows that when she places 80° beer in the refrigerator that is kept at a constant temperature of 40° it cools to 60° in an hour. By what time should she put the beer in the refrigerator to ensure that it will be ready for her friends?

1.3 Linear First Order Equations

A **linear** first order differential equation is an equation of the form

$$y' + p(t)y = f(t). \tag{1}$$

The primary objects of study in the current section are the linear first order differential equations where the **coefficient function** p and the **forcing function** f are continuous functions from an interval I into $\mathbb{R}$. In some exercises and in some later sections of the text, we shall have occasion to consider linear first order differential equations in which the forcing function f is not necessarily continuous, but for now we restrict ourselves to the case where both p and f are continuous. Equation (1) is **homogeneous** if no forcing function is present; i.e., if $f(t) = 0$ for *all* $t \in I$; the equation is **inhomogeneous** if the forcing function f is not 0, i.e., if $f(t) \neq 0$ for *some* $t \in I$. Equation (1) is **constant coefficient** provided the coefficient function p is a constant function, i.e., $p(t) = p_0 \in \mathbb{R}$ for all $t \in I$.

Example 1.3.1. Consider the following list of first order differential equations.

1. $y' = y - t$
2. $y' + ty = 0$
3. $y' = f(t)$
4. $y' + y^2 = t$
5. $ty' + y = t^2$
6. $y' - \frac{3}{t}y = t^4$
7. $y' = 7y$

All of these equations except for $y' + y^2 = t$ are linear. The presence of the y^2 term prevents this equation from being linear. The second and the last equation are homogeneous, while the first, third, fifth and sixth equations are inhomogeneous. The first, third, and last equation are constant coefficient, with $p(t) = -1$, $p(t) = 0$, and $p(t) = -7$ respectively. For the fifth and sixth equations, the interval I on which the coefficient function $p(t)$ and forcing function $f(t)$ are continuous can be either $(-\infty, 0)$ or $(0, \infty)$. In both of these cases, $p(t) = 1/t$ or $p(t) = -3/t$ fails to be continuous at $t = 0$. For the first, second, and last equations, the interval I is all of $\mathbb{R}$, while for the third equation I is any interval on which the forcing function $f(t)$ is continuous. Note that only the second, third and last equations are separable. □

Remark 1.3.2. Notice that Equation (1), which is the traditional way to express a linear first order differential equation, is not in the standard form of Equation (1). In standard form, Equation (1) becomes

$$y' = -p(t)y + f(t), \tag{2}$$

so that the function $F(t, y)$ of Equation (1) is $F(t, y) = -p(t)y + f(t)$. The standard form of the equation is useful for expressing the hypotheses which will be used in the existence and uniqueness results of Section 1.5, while the form given by Equation (1) is particularly useful for describing the solution algorithm to be presented in this section. From Equation (2) one sees that if a first order linear equation is homogeneous (i.e. $f(t) = 0$ for all t), then the equation is separable (the right hand side is $-p(t)y$) and the technique of the previous section applies, while if neither $p(t)$ nor $f(t)$ is the zero function, then Equation (2) is *not* separable, and hence the technique of the previous section is not applicable.

We will describe an algorithm for finding all solutions to the linear differential equation

$$y' + p(t)y = f(t)$$

which is based on first knowing how to solve homogeneous linear equations (i.e., $f(t) = 0$ for all t). But, as we observed above, the homogeneous linear equation is separable, and hence we know how to solve it.

Homogeneous Linear Equation: $y' = h(t)y$

Since the equation $y' = h(t)y$ is separable, we first separate the variables and write the equation in differential form:

$$\frac{1}{y}\,dy = h(t)\,dt. \tag{$*$}$$

If $H(t) = \int h(t)\,dt$ is any antiderivative of $h(t)$, then integration of both sides of Equation $(*)$ gives

$$\ln|y| = H(t) + c$$

where c is a constant of integration. Applying the exponential function to both sides of this equation gives

$$|y| = e^{\ln|y|} = e^{H(t)+c} = e^c e^{H(t)}.$$

Since c is an arbitrary constant, e^c is an arbitrary *positive* constant. Then $y = \pm|y| = \pm e^c e^{H(t)}$ where $\pm e^c$ will be an arbitrary nonzero constant, which, as usual we will continue to denote by c. Since the constant function $y(t) = 0$ is also a solution to $(*)$, and we conclude that, if $H(t) = \int h(t)\,dt$, then the general solution to $y' = h(t)y$ is

$$\boxed{y(t) = ce^{H(t)}} \tag{3}$$

where c denotes any real number.

Example 1.3.3. Solve the equation $y' = \dfrac{3}{t}y$ on the interval $(0, \infty)$.

► **Solution.** In this case $h(t) = \dfrac{3}{t}$ so that an antiderivative on the interval $(0, \infty)$ is

$$H(t) = \int \frac{3}{t}\,dt = 3\ln t = \ln(t^3).$$

Hence then general solution of $y' = \dfrac{3}{t}y$ is

$$y(t) = ce^{H(t)} = ce^{\ln(t^3)} = ct^3.$$

◄

We can now use the homogeneous case to transform an arbitrary first order linear differential equation into an equation which can be solved by antidifferentiation. What results is an algorithmic procedure for determining all solutions to the linear first order equation

$$y' + p(t)y = f(t). \tag{†}$$

The key observation is that the left hand side of this equation looks *almost* like the derivative of a product. Recall that if $z(t) = \mu(t)y(t)$, then

$$z'(t) = \mu(t)y'(t) + \mu'(t)y(t). \tag{‡}$$

Comparing this with Equation (†), we see that what is missing is the coefficient $\mu(t)$ in front of $y'(t)$. If we multiply Equation (†) by $\mu(t)$, we get an equation

$$\mu(t)y'(t) + \mu(t)p(t)y(t) = \mu(t)f(t).$$

The left hand side of this equation agrees with the right hand side of (‡) provided the multiplier function $\mu(t)$ is chosen so that the coefficients of $y(t)$ agree in both equations. That is, choose $\mu(t)$, if possible, so that

$$\mu'(t) = p(t)\mu(t).$$

But this is a homogeneous linear first order differential equation, so by Equation (3) we may take $\mu(t) = e^{P(t)}$ where $P(t)$ is any antiderivative of $p(t)$ on the given interval I. The function $\mu(t)$ is known an an **integrating factor** for the equation $y' + p(t)y = f(t)$, since after multiplication by $\mu(t)$, the left hand side becomes a derivative $(\mu(t)y)'$ and the equation itself becomes

$$(\mu(t)y)' = \mu(t)f(t),$$

which is an equation that can be solved by integration. Recalling that $\int g'(t)\,dt = g(t)+c$, we see that integrating the above equation gives

$$\mu(t)y(t) = \int \mu(t)f(t)\,dt.$$

Putting together all of our steps, we arrive at the following theorem describing all the solutions of a first order linear differential equation. The proof is nothing more than an explicit codification of the steps delineated above into an algorithm to follow.

Theorem 1.3.4. *Let $p(t)$, $f(t)$ be continuous functions on an interval I. A function $y(t)$ is a solution of of the first order linear differential equation $y' + p(t)y = f(t)$ (Equation (1)) on I if and only if*

$$\boxed{y(t) = ce^{-P(t)} + e^{-P(t)} \int e^{P(t)} f(t)\,dt} \tag{4}$$

for all $t \in I$, where $c \in \mathbb{R}$, and $P(t)$ is some antiderivative of $p(t)$ on the interval I.

Proof. Let $y(t) = ce^{-P(t)} + e^{-P(t)} \int e^{P(t)} f(t)\,dt$. Since $P'(t) = p(t)$ and $\frac{d}{dt} \int e^{P(t)} f(t)\,dt = e^{P(t)} f(t)$ (this is what it means to be an antiderivative of $e^{P(t)} f(t)$) we obtain

$$\begin{aligned} y'(t) &= -cp(t)e^{-P(t)} - p(t)e^{-P(t)} \int e^{P(t)} f(t)\,dt + e^{-P(t)} e^{P(t)} f(t) \\ &= -p(t)\left(ce^{-P(t)} + e^{-P(t)} \int e^{P(t)} f(t)\,dt\right) + f(t) \\ &= -p(t)y(t) + f(t) \end{aligned}$$

for all $t \in I$. This shows that every function of the form (4) is a solution of Equation (1). Next we show that any solution of Equation (1) has a representation in the form of Equation (4). This is essentially what we have already done in the paragraphs prior to the statement of the theorem. What we shall do now is summarize the steps to be taken to implement this algorithm. Let $y(t)$ be a solution of Equation (1) on the interval I. Then we perform the following step-by-step procedure, which will be crucial when dealing with concrete examples.

Algorithm 1.3.5 (Solution of First Order Linear Equations). Follow the following procedure to put any solution $y(t)$ of Equation (1) into the form given by Equation (4).

1. Compute an antiderivative $P(t) = \int p(t)\,dt$ and multiply the equation $y' + p(t)y = f(t)$ by the **integrating factor** $\mu(t) = e^{P(t)}$. This yields

   (I) $$e^{P(t)}y'(t) + p(t)e^{P(t)}y(t) = e^{P(t)}f(t).$$

2. The function $\mu(t) = e^{P(t)}$ is an integrating factor (see the paragraphs prior to the theorem) which means that the left hand side of Equation (I) is a perfect derivative, namely $(\mu(t)y(t))'$. Hence, Equation (I) becomes

   (II) $$\frac{d}{dt}(\mu(t)y(t)) = e^{P(t)}f(t).$$

3. Now we take an antiderivative of both sides and observe that they must coincide up to a constant $c \in \mathbb{R}$. This yields

   (III) $$e^{P(t)}y(t) = \int e^{P(t)}f(t)\,dt + c.$$

4. Finally, multiply by $\mu(t)^{-1} = e^{-P(t)}$ to get that $y(t)$ is of the form

   (IV) $$y(t) = ce^{-P(t)} + e^{-P(t)} \int e^{P(t)}f(t)\,dt.$$

This shows that any solution of Equation (1) is of the form given by Equation (4), and moreover, the steps of Algorithm 1.3.5 tell one precisely how to find this form. □

Remark 1.3.6. You should **not** memorize formula (4). What you should remember instead is the sequence of steps in Algorithm 1.3.5, and apply these steps to each concretely presented linear first order differential equation (given in the form of Equation (1)). To summarize the algorithm in words:

1. Find an integrating factor $\mu(t)$.
2. Multiply the equation by $\mu(t)$, insuring that the left hand side of the equation is a perfect derivative.
3. Integrate both sides of the resulting equation.
4. Divide by $\mu(t)$ to give the solution $y(t)$.

Example 1.3.7. Find all solutions of the differential equation $t^2y' + ty = 1$ on the interval $(0, \infty)$.

► **Solution.** Clearly, you could bring the equation into the standard form of Equation (1), that is

$$y' + \frac{1}{t}y = \frac{1}{t^2},$$

identify $p(t) = \frac{1}{t}$ and $f(t) = \frac{1}{t^2}$, compute an antiderivative $P(t) = \ln(t)$ of $p(t)$ on the interval $(0, \infty)$, plug everything into formula (4), and then compute the resulting integral. This is a completely valid procedure if you are good in memorizing formulas. Since we are not good at memorization, we prefer go through the steps of Algorithm 1.3.5 explicitly.

First bring the differential equation into the standard form

$$y' + \frac{1}{t}y = \frac{1}{t^2}.$$

Then compute an antiderivative $P(t)$ of the function in front of y and multiply the equation by the integrating factor $\mu(t) = e^{P(t)}$. In our example, we take $P(t) = \ln(t)$ and multiply the equation by $\mu(t) = e^{P(t)} = e^{\ln(t)} = t$ (we could also take $P(t) = \ln(t) + c$ for any constant c, but the computations are easiest if we set the constant equal to zero). This yields

(I) $$ty' + y = \frac{1}{t}.$$

Next observe that the left side of this equality is equal to $\frac{d}{dt}(ty)$ (see Step 2 of Algorithm 1.3.5). Thus,

(II) $$\frac{d}{dt}(ty) = \frac{1}{t}.$$

Now take antiderivatives of both sides and observe that they must coincide up to a constant $c \in \mathbb{R}$. Thus,

(III) $$ty = \ln(t) + c, \quad \text{or}$$

(IV) $$y(t) = c\frac{1}{t} + \frac{1}{t}\ln(t).$$

Observe that $y_h(t) = c\frac{1}{t}$ ($c \in \mathbb{R}$) is the general solution of the homogeneous equation $t^2y' + ty = 0$, and that $y_p(t) = \frac{1}{t}\ln(t)$ is a particular solution of $t^2y' + ty = 1$. Thus, all solutions are given by $y(t) = y_h(t) + y_p(t)$. As the following remark shows, this holds for all linear first order differential equations. ◀

Remark 1.3.8. Analyzing the general solution $y(t) = ce^{-P(t)} + e^{-P(t)} \int e^{P(s)} f(s)\, ds$, we see that this general solution is the sum of two parts. Namely, $y_h(t) = ce^{-P(t)}$ which is the general solution of the homogeneous problem

$$y' + p(t)y = 0,$$

and $y_p(t) = e^{-P(t)} \int e^{P(s)} f(s)\, ds$ which is a particular, i.e., a single, solution of the inhomogeneous problem

$$y' + p(t)y = f(t).$$

The homogeneous equation $y' + p(t)y = 0$ is known as the **associated homogeneous equation** of the linear equation $y' + p(t)y = f(t)$. That is, the right hand side of the general linear equation is replaced by 0 to get the associated homogeneous equation. The relationship between the *general* solution $y_g(t)$ of $y' + p(t)y = f(t)$, a *particular* solution $y_p(t)$ of this equation, and the *general* solution $y_h(t)$ of the associated homogeneous equation $y' + p(t)y = 0$, is usually expressed as

$$y_g(t) = y_h(t) + y_p(t). \tag{5}$$

What this means is that *every* solution to $y' + p(t)y = f(t)$ can be obtained by starting with a single solution $y_p(t)$ and adding to that an appropriate solution of $y' + p(t)y = 0$. The key observation is the following. Suppose that $y_1(t)$ and $y_2(t)$ are *any* two solutions of $y' + p(t)y = f(t)$. Then

$$\begin{aligned} (y_2 - y_1)'(t) + p(t)(y_2 - y_1)(t) &= (y_2'(t) + p(t)y_2(t)) - (y_1'(t) + p(t)y_1(t)) \\ &= f(t) - f(t) \\ &= 0, \end{aligned}$$

so that $y_2(t) - y_1(t)$ is a solution of the associated homogeneous equation $y' + p(t)y = 0$, and $y_2(t) = y_1(t) + (y_2(t) - y_1(t))$. Therefore, given a solution $y_1(t)$ of $y' + p(t)y = f(t)$, any other solution $y_2(t)$ is obtained from $y_1(t)$ by adding a solution (specifically $y_2(t) - y_1(t)$) of the associated homogeneous equation $y' + p(t)y = 0$.

This observation is a general property of solutions of **linear** equations, whether they are differential equations of first order (as above), differential equations of higher order (to be studied in Chapter 3), linear algebraic equations, or linear equations $L(\mathbf{y}) = \mathbf{f}$ in any **vector space**, which is the mathematical concept created to handle the features common to problems of linearity.. Thus, the general solution set $\mathcal{S} = \mathbf{y}_g$ of any linear equation $L(\mathbf{y}) = \mathbf{f}$ is of the form

$$\mathbf{y}_g = \mathcal{S} = L^{-1}(0) + \mathbf{y}_p = \mathbf{y}_h + \mathbf{y}_p,$$

where $L(\mathbf{y}_p) = \mathbf{f}$ and $L^{-1}(0) = \mathbf{y}_h = \{\mathbf{y} : L(\mathbf{y}) = 0\}$.

Corollary 1.3.9. *Let $p(t)$, $f(t)$ be continuous on an interval I, $t_0 \in I$, and $y_0 \in \mathbb{R}$. Then the unique solution of the* **initial value problem**

$$y' + p(t)y = f(t), \quad y(t_0) = y_0 \tag{6}$$

is given by

$$\boxed{y(t) = y_0 e^{-P(t)} + e^{-P(t)} \int_{t_0}^{t} e^{P(u)} f(u)\, du,} \tag{7}$$

where $P(t) = \int_{t_0}^{t} p(u)\, du$.

Proof. Since $P(t)$ is an antiderivative of $p(t)$, we see that $y(t)$ has the form of Equation (4), and hence Theorem 1.3.4 guarantees that $y(t)$ is a solution of the linear first order equation $y' + p(t)y = f(t)$. Moreover, $P(t_0) = \int_{t_0}^{t_0} p(u)\, du = 0$, and

$$y(t_0) = y_0 e^{-P(t_0)} + e^{-P(t_0)} \int_{t_0}^{t_0} e^{P(u)} f(u)\, du = y_0,$$

so that $y(t)$ is a solution of the initial value problem (6). Suppose that $y_1(t)$ is any other solution of Equation (6). Then $y_2(t) := y(t) - y_1(t)$ is a solution of the associated homogeneous equation

$$y' + p(t)y = 0, \quad y(t_0) = 0.$$

It follows from Equation (3) that $y_2(t) = ce^{-\tilde{P}(t)}$ for some constant $c \in \mathbb{R}$ and an antiderivative $\tilde{P}(t)$ of $p(t)$. Since $y_2(t_0) = 0$ and $e^{-\tilde{P}(t_0)} \neq 0$, it follows that $c = 0$. Thus, $y(t) - y_1(t) = y_2(t) = 0$ for all $t \in I$. This shows that $y_1(t) = y(t)$ for all $t \in I$, and hence $y(t)$ is the only solution of Equation (6). □

Example 1.3.10. Find the solution of the initial value problem $y' = -ty + t, \quad y(2) = 7$ on $\mathbb{R}$.

▶ **Solution.** Again, you could bring the differential equation into the standard form

$$y' + ty = t,$$

identify $p(t) = t$ and $f(t) = t$, compute the antiderivative

$$P(t) = \int_2^t u\, du = \frac{t^2}{2} - 2$$

of $p(t)$, plug everything into the formula (4), and then compute the integral in (7) to get

$$\begin{aligned} y(t) &= y_0 e^{-P(t)} + e^{-P(t)} \int_{t_0}^{t} e^{P(u)} f(u)\, du \\ &= 7e^{\frac{-t^2}{2}+2} + e^{\frac{-t^2}{2}+2} \int_{2}^{t} u e^{\frac{u^2}{2}-2}\, du. \end{aligned}$$

However, we again prefer to follow the steps of the algorithm. First we proceed as in Example 1.3.7 and find the general solution of

$$y' + ty = t.$$

To do so we multiply the equation by the integrating factor $e^{t^2/2}$ and obtain

$$e^{t^2/2} y' + t e^{t^2/2} y = t e^{t^2/2}.$$

Since the left side is the derivative of $e^{t^2/2} y$, this reduces to

$$\frac{d}{dt}\left(e^{t^2/2} y\right) = t e^{t^2/2}.$$

Since $e^{t^2/2}$ is the antiderivative of $te^{t^2/2}$, it follows that

$$e^{t^2/2} y(t) = e^{t^2/2} + c, \quad \text{or} \quad y(t) = ce^{-t^2/2} + 1.$$

Finally, we determine the constant c such that $y(2) = 7$. This yields $7 = ce^{-2} + 1$ or $c = 6e^2$. Thus, the solution is given by

$$y(t) = 6e^{-\frac{t^2}{2}+2} + 1.$$

◀

Corollary 1.3.11. *Let $f(t)$ be a continuous function on an interval I and $p \in \mathbb{R}$. Then all solution of the first order, inhomogeneous, linear,* **constant coefficient** *differential equation*

$$y' + py = f(t)$$

are given by

$$y(t) = ce^{-pt} + \int e^{-p(t-u)} f(u)\, du.$$

Moreover, for any $t_0, y_0 \in \mathbb{R}$, the unique solution of the initial value problem

$$y' + py = f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = y_0 e^{-p(t-t_0)} + \int_{t_0}^{t} e^{-p(t-u)} f(u)\, du.$$

Proof. The statements follow immediately from Corollary 1.3.9. □

Example 1.3.12. Find the solution of the initial value problem $y' = -y + 4, \quad y(0) = 8$ on $\mathbb{R}$.

▶ **Solution.** We write the equation as $y' + y = 4$ and apply Corollary 1.3.11. This yields

$$y(t) = 8e^{-t} + \int_0^t 4e^{-(t-u)}\,ds = 8e^{-t} + 4e^{-t}\int_0^t e^u\,du = 8e^{-t} + 4e^{-t}\left[e^t - 1\right] = 4e^{-t} + 4.$$

◀

Example 1.3.13. Find the solution of the initial value problem $y' + y = \frac{1}{1-t}, \quad y(0) = 0$ on the interval $(-\infty, 1)$.

▶ **Solution.** By Corollary 1.3.11, $y(t) = e^{-t}\int_0^t \frac{1}{1-u}e^u\,du$. Since the function $\dfrac{1}{1-u}e^u$ is not integrable in closed form on the interval $(-\infty, 1)$, we might be tempted to stop at this point and say that we have solved the equation. While this is a legitimate statement, the present representation of the solution is of little practical use and a further detailed study is necessary if you are "really" interested in the solution. Any further analysis (numerical calculations, qualitative analysis, etc.) would be based on what type of information you are attempting to ascertain about the solution. ◀

We can use our analysis of first order linear differential equations to solve the mixing problem set up in Example 1.1.9. For convenience we restate the problem.

Example 1.3.14. Consider a tank that contains 2000 gallons of water in which 10 lbs of salt are dissolved. Suppose that a water-salt mixture containing 0.1 lb/gal enters the tank at a rate of 2 gal/min, and assume that the well-stirred mixture flows from the tank at the same rate of 2 gal/min. Find the amount $y(t)$ of salt (expressed in pounds) which is present in the tank at all times t measured in minutes.

▶ **Solution.** In Example 1.1.9, it was determined that $y(t)$ satisfies the initial value problem

$$y' + (0.001)y = 0.2, \quad y(0) = 10. \tag{8}$$

This equation has an integrating factor $\mu(t) = e^{(0.001)t}$, so multiplying the equation by $\mu(t)$ gives

$$\left(e^{(0.001)t}y\right)' = (0.2)e^{(0.001)t}.$$

Integration of this equation gives $e^{(0.001)t}y = 200e^{(0.001)t} + c$, or after solving for y,

$$y(t) = 200 + ce^{-(0.001)t}.$$

Setting $t = 0$ gives $10 = y(0) = 200 + c$ so that $c = -190$ and the final answer is

$$y(t) = 200 - 190e^{-(0.001)t}.$$

◀

Next we consider a numerical example of the general mixing problem considered in Example 1.1.10

Example 1.3.15. A large tank contains 100 gal of brine in which 50 lb of salt is dissolved. Brine containing 2 lb of salt per gallon runs into the tank at the rate of 6 gal/min. The mixture, which is kept uniform by stirring, runs out of the tank at the rate of 4 gal/min. Find the amount of salt in the tank at the end of t minutes.

▶ **Solution.** Let $y(t)$ denote the number of pounds of salt in the tank after t minutes; note that the tank will contain $100 + (6 - 4)t$ gallons of brine at this time. The concentration (number of pounds per gallon) will then be

$$\frac{y(t)}{100 + 2t} \text{ lb/gal.}$$

Instead of trying to find the amount (in pounds) of salt $y(t)$ at time t directly, we will follow the analysis of Example 1.1.10 and determine the rate of change of $y(t)$, i.e., $y'(t)$. But the the change of $y(t)$ at time t is governed by the principle

$$y'(t) = \text{ input rate } - \text{ output rate},$$

where all three rates have to be measured in the same unit, which we take to be lb/min. Thus,

$$\begin{aligned} \text{input rate } &= 2\,\text{lb/gal} \times 6 \text{ gal/min} = 12 \text{ lb/min}, \\ \text{output rate } &= \frac{y(t)}{100 + 2t} \text{ lb/gal} \times \; 4 \text{ gal/min } = \frac{4y(t)}{100 + 2t} \text{ lb/min.} \end{aligned}$$

This yields the initial value problem

$$y'(t) = 12 - \frac{4y(t)}{100 + 2t}, \quad y(0) = 50$$

which can be solved as in the previous examples. The solution is seen to be

$$y(t) = 2(100 + 2t) - \frac{15(10^5)}{(100 + 2t)^2}.$$

After 50 min, for example, there will be 362.5 lb of salt in the tank and 200 gal of brine. ◀

Exercises

Find the general solution of the given differential equation. If an initial condition is given, find the particular solution which satisfies this initial condition. Examples 1.3.3, 1.3.7, and 1.3.10 are relevant examples to review, and detailed solutions of a few of the exercises will be provided for you to study.

1. $y'(t) + 3y(t) = e^t$, $y(0) = -2$.

   ▶ **Solution.** This equation is already in standard form (Equation (3.1.1)) with $p(t) = 3$. An antiderivative of $p(t)$ is $P(t) = \int 3\,dt = 3t$. If we multiply the differential equation $y'(t) + 3y(t) = e^t$ by $P(t)$, we get the equation

   $$e^{3t}y'(t) + 3e^{3t}y(t) = e^{4t},$$

   and the left hand side of this equation is a perfect derivative, namely, $\frac{d}{dt}(e^{3t}y(t))$. Thus,

   $$\frac{d}{dt}(e^{3t}y(t)) = e^{4t}.$$

   Now take antiderivatives of both sides and observe that they must coincide up to a constant $c \in \mathbb{R}$. This gives

   $$e^{3t}y(t) = \frac{1}{4}e^{4t} + c.$$

   Now, multiplying by e^{-3t} gives

   $$\boxed{y(t) = \frac{1}{4}e^t + ce^{-3t}} \qquad (*)$$

   for the general solution of the equation $y'(t) + 3y(t) = e^t$. To choose the constant c to satisfy the initial condition $y(0) = -2$, substitute $t = 0$ into Equation (*) to get $-2 = y(0) = \frac{1}{4} + c$ (remember that $e^0 = 1$). Hence $c = -\frac{9}{4}$, and the solution of the initial value problem is

   $$\boxed{y(t) = \frac{1}{4}e^t - \frac{9}{4}e^{-3t}.}$$

   ◀

2. $(\cos t)y'(t) + (\sin t)y(t) = 1$, $y(0) = 5$

▶ **Solution.** Divide the equation by $\cos t$ to put it in the standard form

$$y'(t) + (\tan t)y(t) = \sec t.$$

In this case $p(t) = \tan t$ and an antiderivative is $P(t) = \int \tan t\, dt = \ln(\sec t)$. (We do not need $|\sec t|$ since we are working near $t = 0$ where $\sec t > 0$.) Now multiply the differential equation $y'(t) + (\tan t)y(t) = \sec t$ by $e^{P(t)} = e^{\ln \sec t} = \sec t$ to get $(\sec t)y'(t) + (\sec t \tan t)y(t) = \sec^2 t$, the left hand side of which is a perfect derivative, namely $\frac{d}{dt}((\sec t)y(t))$. Thus

$$\frac{d}{dt}((\sec t)y(t)) = \sec^2 t$$

and taking antiderivatives of both sides gives

$$(\sec t)y(t) = \tan t + c$$

where $c \in \mathbb{R}$ is a constant. Now multiply by $\cos t$ to eliminate the $\sec t$ in front of $y(t)$, and we get

$$\boxed{y(t) = \sin t + c \cos t}$$

for the general solution of the equation, and letting $t = 0$ gives $5 = y(0) = \sin 0 + c \cos 0 = c$ so that the solution of the initial value problem is

$$\boxed{y(t) = \sin t + 5 \cos t.}$$

◀

3. $y' - 2y = e^{2t}$, $y(0) = 4$

4. $y' - 2y = e^{-2t}$, $y(0) = 4$

5. $ty' + y = e^t$, $y(1) = 0$

6. $ty' + y = e^{2t}$, $y(1) = 0$.

7. $y' = (\tan t)y + \cos t$

8. $y' + ty = 1$, $y(0) = 1$.

9. $ty' + my = t \ln(t)$, where m is a constant.

10. $y' = -\frac{y}{t} + \cos(t^2)$

11. $t(t+1)y' = 2 + y$.

12. $y' + ay = b$, where a and b are constants.

13. $y' + y\cos t = \cos t$, $y(0) = 1$

14. $y' - \dfrac{2}{t+1}y = (t+1)^2$

15. $y' - \dfrac{2}{t}y = \dfrac{t+1}{t}$, $y(1) = -3$

16. $y' + ay = e^{-at}$, where a is a constant.

17. $y' + ay = e^{bt}$, where a and b are constants and $b \neq -a$.

18. $y' + ay = t^n e^{-at}$, where a is a constant.

19. $y' = y\tan t + \sec t$

20. $ty' + 2y\ln t = 4\ln t$

21. $y' - \dfrac{n}{t}y = e^t t^n$

22. $y' - y = te^{2t}$, $y(0) = a$

23. $ty' + 3y = t^2$, $y(-1) = 2$

24. $t^2 y' + 2ty = 1$, $y(2) = a$

Before attempting the following exercises, you may find it helpful to review the examples in Section 1.1 related to mixing problems.

25. A tank contains 10 gal of brine in which 2 lb of salt are dissolved. Brine containing 1 lb of salt per gallon flows into the tank at the rate of 3 gal/min, and the stirred mixture is drained off the tank at the rate of 4 gal/min. Find the amount $y(t)$ of salt in the tank at any time t.

26. A 100 gal tank initially contains 10 gal of fresh water. At time $t = 0$, a brine solution containing .5 lb of salt per gallon is poured into the tank at the rate of 4 gal/min while the well-stirred mixture leaves the tank at the rate of 2 gal/min.

(a) Find the time T it takes for the tank to overflow.

(b) Find the amount of salt in the tank at time T.

(c) If $y(t)$ denotes the amount of salt present at time t, what is $\lim_{t\to\infty} y(t)$?

27. A tank contains 100 gal of brine made by dissolving 80 lb of salt in water. Pure water runs into the tank at the rate of 4 gal/min, and the mixture, which is kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t. Find the concentration of salt in the tank at any time t.

28. For this problem, our tank will be a lake and the brine solution will be polluted water entering the lake. Thus assume that we have a lake with volume V which is fed by a polluted river. Assume that the rate of water flowing into the lake and the rate of water flowing out of the lake are equal. Call this rate r, let c be the concentration of pollutant in the river as it flows *into* the lake, and assume perfect mixing of the pollutant in the lake (this is, of course, a very unrealistic assumption).

    (a) Write down and solve a differential equation for the amount $P(t)$ of pollutant in the lake at time t and determine the limiting *concentration* of pollutant in the lake as $t \to \infty$.

    (b) At time $t = 0$, the river is cleaned up, so no more pollutant flows into the lake. Find expressions for how long it will take for the pollution in the lake to be reduced to (i) 1/2 (ii) 1/10 of the value it had at the time of the clean-up.

    (c) Assuming that Lake Erie has a volume V of 460 km^3 and an inflow-outflow rate of $r = 175$ km^3/year, give numerical values for the times found in Part (b). Answer the same question for Lake Ontario, where it is assumed that $V = 1640$ km^3 and $r = 209$ km^3/year.

29. A 30 liter container initially contains 10 liters of pure water. A brine solution containing 20 grams salt per liter flows into the container at a rate of 4 liters per minute. The well stirred mixture is pumped out of the container at a rate of 2 liters per minute.

    (a) How long does it take the container to overflow?

    (b) How much salt is in the tank at the moment the tank begins to overflow?

30. A tank holds 10 liters of pure water. A brine solution is poured into the tank at a rate of 1 liter per minute and kept well stirred. The mixture leaves the tank at the same rate. If the brine solution has a concentration of 1 kg salt per liter what will the concentration be in the tank after 10 minutes.

1.4 Direction Fields

The geometric interpretation of the derivative of a function $y(t)$ at t_0 as the slope of the tangent line to the graph of $y(t)$ at $(t_0,\ y(t_0))$ provides us with an elementary and often very effective method for the visualization of the **solution curves** (:= graphs of solutions) for a first order differential equation. The visualization process involves the construction of what is known as a **direction field** or **slope field** for the differential equation. For this construction we proceed as follows.

Construction of Direction Fields

(1) If the equation is not already in standard form (Equation (1)) solve the equation for y' to put it in the standard form $y' = F(t, y)$.

(2) Choose a grid of points in a rectangle $\mathcal{R} = \{(t, y) : a \le t \le b;\ c \le y \le d\}$ in the (t, y)-plane.

(3) At each grid point (t, y), the number $F(t, y)$ represents the slope of a solution curve through this point; for example if $y' = y^2 - t$ so that $F(t, y) = y^2 - t$, then at the point $(1, 1)$ the slope is $F(1, 1) = 1^2 - 1 = 0$, at the point $(2, 1)$ the slope is $F(2, 1) = 1^2 - 2 = -1$, and at the point $(1, -2)$ the slope is $F(1, -2) = 3$.

(4) Through the point (t, y) draw a small line segment having the slope $F(t, y)$. Thus, for the equation $y' = y^2 - t$, we would draw a small line segment of slope 0 through $(1, 1)$, slope -1 through $(2, 1)$ and slope 3 through $(1, -2)$. With a graphing calculator, one of the computer mathematics programs Maple, Mathematica or MATLAB (which we refer to as the three M's) [1], or with pencil, paper, and a lot of patience, you can draw many such line segments. The resulting picture is called a **direction field** for the differential equation $y' = F(t, y)$.

(5) With some luck with respect to scaling and the selection of the (t, y)-rectangle $\mathcal{R}$, you will be able to visualize some of the line segments running together to make a graph of one of the solution curves.

(6) To sketch a solution curve of $y' = F(t, y)$ from a direction field, start with a point $P_0 = (t_0, y_0)$ on the grid, and sketch a short curve through P_0 with tangent slope $F(t_0, y_0)$. Follow this until you are at or close to another grid point $P_1 = (t_1, y_1)$. Now continue the curve segment by using the updated tangent slope $F(t_1, y_1)$.

[1] We have used the Student Edition of MATLAB, Version 6, and the functions dfield6 and pplane6 which we downloaded from the webpage http://math.rice.edu/dfield. To see dfield6 in action, enter dfield6 at the MATLAB prompt

Continue this process until you are forced to leave your sample rectangle $\mathcal{R}$. The resulting curve will be an approximate solution to the initial value problem $y' = F(t, y)$, $y(t_0) = y_0$.

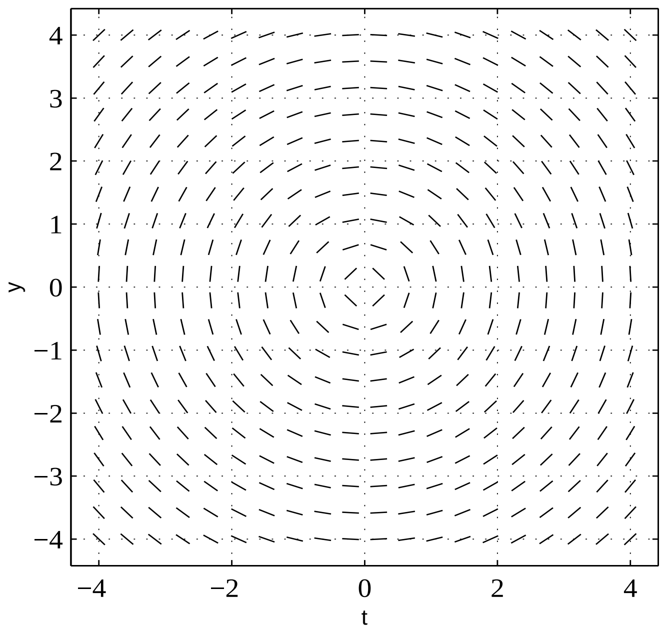


Figure 1.4: Direction Field of $yy' = -t$

Example 1.4.1. Draw the direction field for the differential equation $yy' = -t$. Draw several solution curves on the direction field, then solve the differential equation explicitly and describe the general solution.

► **Solution.** Before we can draw the direction field, it is necessary to first put the differential equation $yy' = -t$ into standard form by solving for y'. Solving for y' gives the equation

$$(*) \qquad y' = -\frac{t}{y}.$$

Notice that this equation is not defined for $y = 0$, even though the original equation is. Thus, we should be alert to potential problems arising from this defect. We have chosen a rectangle $\mathcal{R} = \{(t, y) : -4 \le t,\, y \le 4\}$ for drawing the direction field, and we have chosen to use 20 sample points in each direction, which gives a total of 400 grid points where a slope line will be drawn. Naturally, this is being done by computer (using the

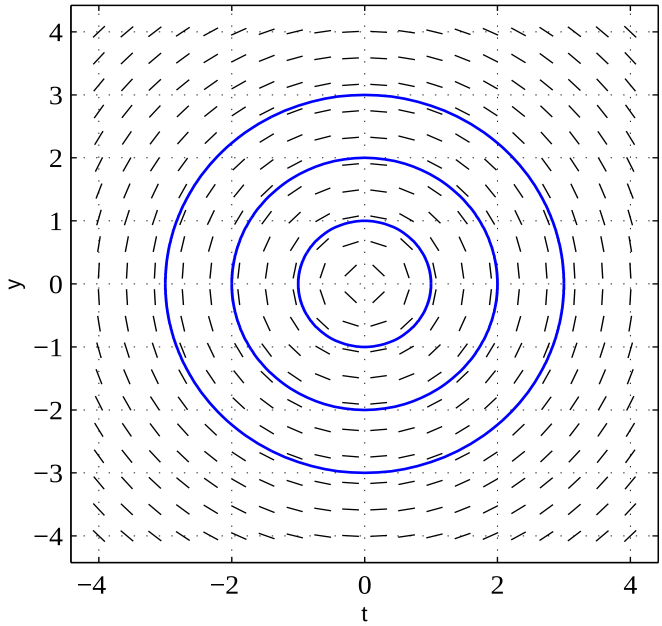


Figure 1.5: Solution Curves for $yy' = -t$

dfield6 tool in MatLab), and not by hand. Figure 1.4 gives the completed direction field, and Figure 1.5 is the same direction field with several solution curves drawn in. The solutions which are drawn in are the solutions of the initial value problems $yy' = -t$, $y(0) = \pm 1, \pm 2, \pm 3$. The solution curves appear to be half circles centered at $(0, 0)$. Since the equation $yy' = -t$ is separable, we can verify that this is in fact true by explicitly solving the equation. Writing the equation in differential form gives $ydy = -tdt$ and integrating gives

$$\frac{y^2}{2} = -\frac{t^2}{2} + c.$$

After multiplying by 2 and renaming the constant, we see that the solutions of $yy' = -t$ are given implicitly by $y^2 + t^2 = c$. Thus, there are two families of solutions of $yy' = -t$, specifically, $y_1(t) = \sqrt{c - t^2}$ (upper semicircle) and $y_2(t) = -\sqrt{c - t^2}$ (lower semicircle). For both families of functions, c is a positive constant and the functions are defined on the interval $(-\sqrt{c}, \sqrt{c})$. For the solutions drawn in Figure 1.5, the constant c is 1, $\sqrt{2}$, and $\sqrt{3}$. Notice that, although y_1 and y_2 are both defined for $t = \pm\sqrt{c}$, they do not satisfy the differential equation at these points since y_1' and y_2' do not exist at these points. Geometrically, this is a reflection of the fact that the circle $t^2 + y^2 = c$ has a vertical tangent at the points $(\pm\sqrt{c}, 0)$ on the t-axis. This is the "defect" that you were warned could occur because the equation $yy' = -t$, when put in standard form

$y' = -t/y$, is not defined for $y = 0$. ◀

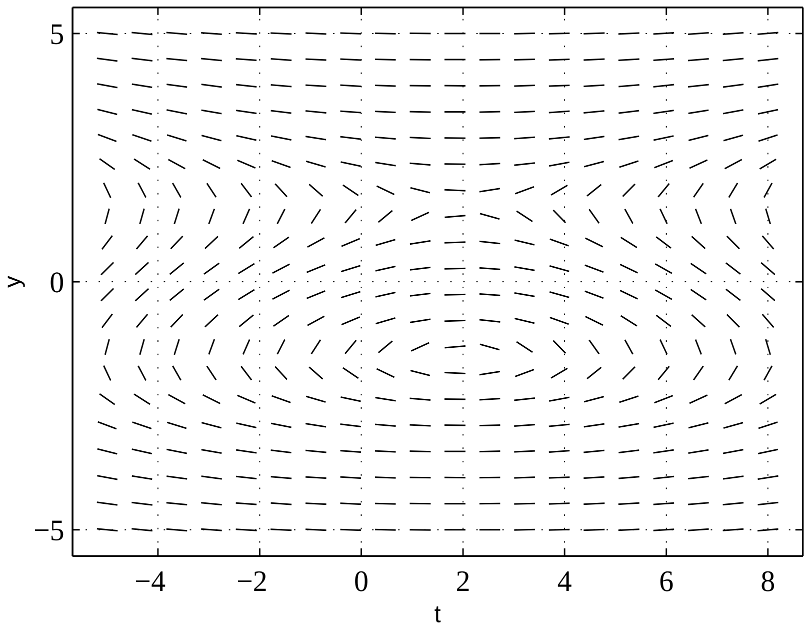


Figure 1.6: Direction Field of $y' = \dfrac{t-2}{3y^2-7}$

It may happen that a formula solution for the differential equation $y' = F(t, y)$ is possible, but the formula is sufficiently complicated that it does not shed much light on the nature of the solutions. In such a situation, it may happen that constructing a direction field and drawing the solution curves on the direction field gives useful insight concerning the solutions. The following example is a situation where the picture is more illuminating than the formula.

Example 1.4.2. Solve the differential equation $y' = \dfrac{t-2}{3y^2-7}$.

▶ **Solution.** The equation is separable, so we proceed as usual by separating the variables, writing the equation in differential form, and then integrating both sides of the equation. In the present case, the differential form of the equation is $(3y^2 - 7)\,dy = (t-2)\,dt$, so that, after integration and clearing denominators, we find that the general solution is given by the implicit equation

$$(*) \qquad 2y^3 - 14y = t^2 - 4t + c.$$

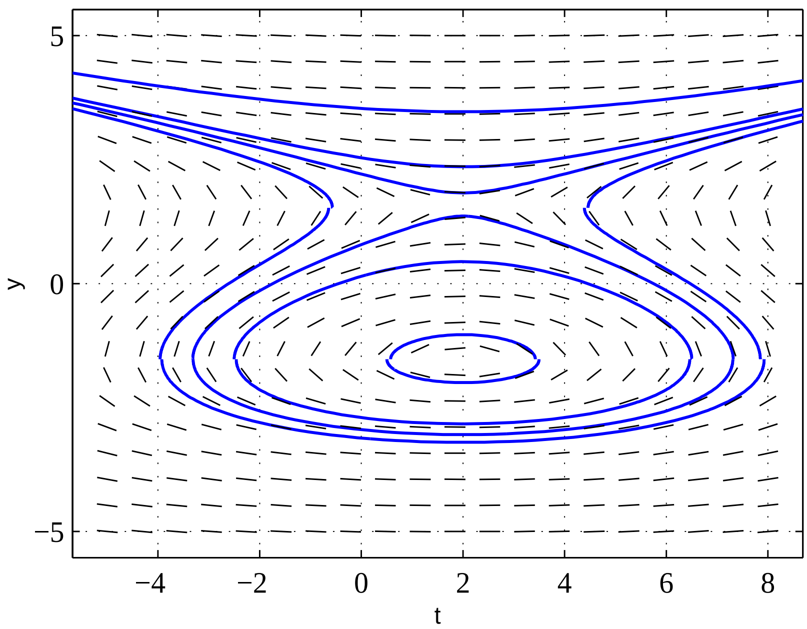


Figure 1.7: Solution Curves for $y' = \dfrac{t-2}{3y^2-7}$

While there is a formula for solving a cubic equation,[2] it is a messy formula which does not necessarily shed great light upon the nature of the solutions as functions of t. However, if we compute the direction field of $y' = \dfrac{t-2}{3y^2-7}$, and use it to draw some solution curves, we see a great deal more concerning the nature of the solutions. Figure 1.6 is the direction field and Figure 1.7 is the direction field with several solutions drawn in. Some observations which can be made from the picture are:

- In the lower part of the picture, the curves seem to be deformed ovals centered about the point $P \approx (2, -1.5)$.

- Above the point $Q \approx (2, 2)$, the curves no longer are closed, but appear to increase indefinitely in both directions.

◀

[2]The formula is known as Cardano's formula after Girolamo Cardano (1501 – 1576), who was the first to publish it.

We conclude our list of examples of direction fields with an example for which the explicit solution formula, found by a method to be considered later, gives even less insight than that considered in the last example. Nevertheless, the direction field and some appropriately chosen solution curves drawn on the direction field, suggest a number of properties of solutions of the differential equation.

Example 1.4.3. The example to be considered is the differential equation

$$(**) \qquad y' = y^2 - t.$$

This equation certainly does not look any more complicated than those considered in previous examples. In fact, the right hand side of this equation is a quadratic which looks simple enough, certainly simpler than the right hand side of the previous example. The parabola $y^2-t=0$ has a particularly simple meaning on the direction field. Namely, every solution of the differential equation $y' = y^2 - t$ which touches the parabola will have a horizontal tangent at that point. That is, for every point $(t_0, y(t_0))$ on the graph of a solution $y(t)$ for which $y(t_0)^2 - t_0 = 0$, we will have $y'(t_0) = 0$. The curve $y^2 - t = 0$ is known as the **nullcline** of the differential equation $y' = y^2 - t$. Figure 1.8 is the direction field for $y' = y^2 - t$. Figure 1.9 shows the solution of the equation $y' = y^2 - t$ which has the initial value $y(0) = 0$, while Figure 1.10 shows a number of different solutions to the equation satisfying various initial conditions $y(0) = y_0$. Unlike the previous examples we have considered, there is no simple formula which gives all of the solutions of $y' = y^2 - t$. There is a formula which involves a family of functions known as Bessel functions. Bessel functions are themselves defined as solutions of a particular second order linear differential equation. For those who are curious, we note that the general solution of $y' = y^2 - t$ is

$$y(t) = \sqrt{t}\,\frac{cK(-\frac{2}{3}, \frac{2}{3}t^{3/2}) - I(-\frac{2}{3}, \frac{2}{3}t^{3/2})}{cK(\frac{1}{3}, \frac{2}{3}t^{3/2}) + I(\frac{1}{3}, \frac{2}{3}t^{3/2})},$$

where

$$I(\mu, z) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+\mu+1)}(z/2)^{2k+\mu}$$

is the modified Bessel function of the first kind, where $\Gamma(x) := \int_0^\infty e^{-t}t^{x-1}\,dt$ denotes the Gamma function, and where

$$K(\mu, z) := \frac{\pi}{2\sin(\mu x)}(I(-\mu, x) - I(\mu, x))$$

is the modified Bessel function of the second kind. [3] As we can see, even if an analytic expression for the general solution of a first order differential equation can be found, it

[3] The solution above can be found easily with symbolic calculators like Maple, Mathematica or

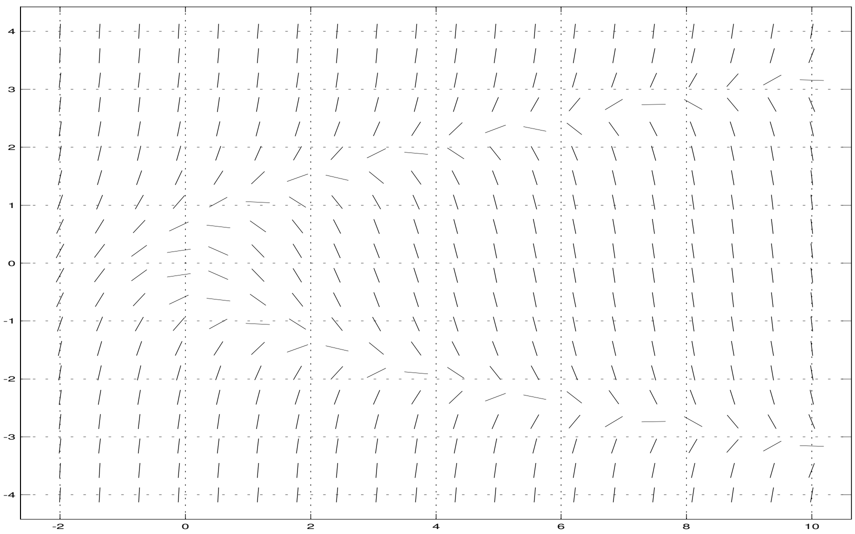


Figure 1.8: Direction Field of $y' = y^2 - t$

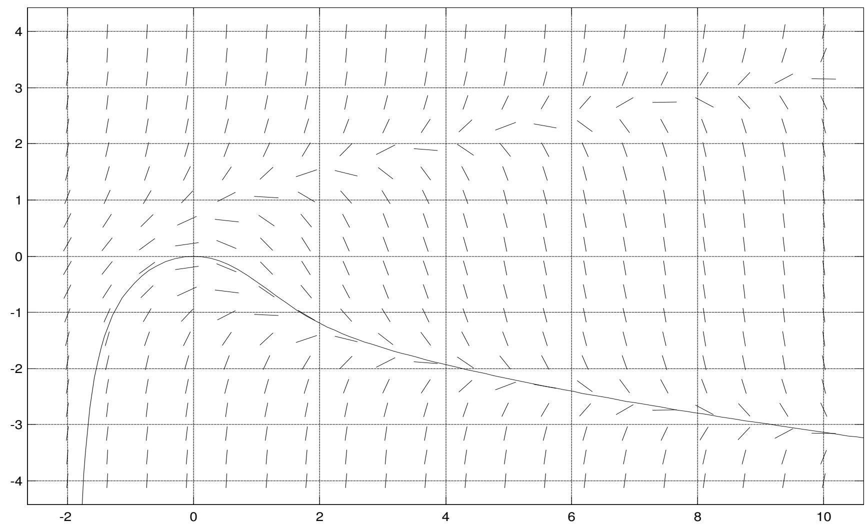


Figure 1.9: The solution curve for $y' = y^2 - t$ with $y(0) = 0$

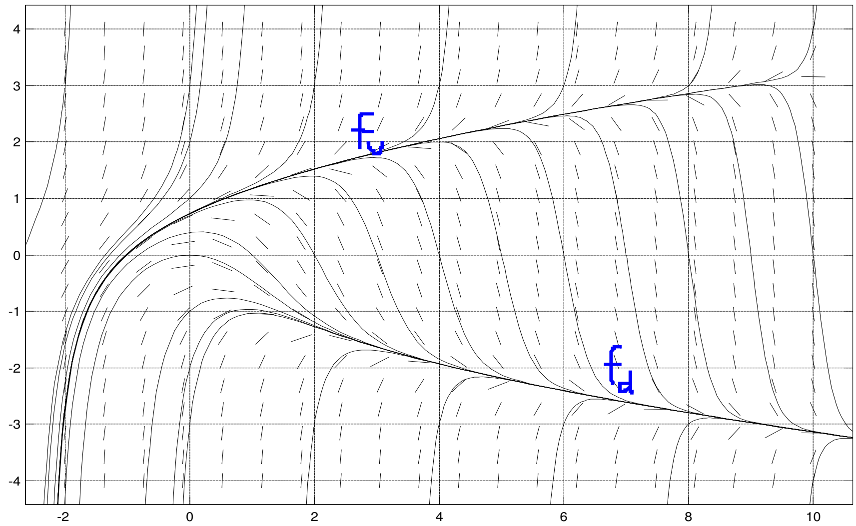


Figure 1.10: Solution curves for $y' = y^2 - t$

might not be very helpful on first sight, and the direction field may give substantially more insight into the true nature of the solutions.

For example, a detailed analysis of the direction field (see Figure 1.10) reveals that the plane seems to be divided into two regions defined by some curve $f_u(t)$. Solution curves going through points above $f_u(t)$ tend towards infinity as $t \to \infty$, whereas solution curves passing through points below $f_u(t)$ seem to approach the solution curve $f_d(t)$ with $y(0) = 0$ as $t \to \infty$.

The equation $y' = y^2 - t$ is an example of a type of differential equation known as a **Riccati equation**. A Ricatti equation is a first order differential equation of the form

$$y' = a(t)y^2 + b(t)y + c(t),$$

where $a(t)$, $b(t)$ and $c(t)$ are continuous functions of t. For more information on this important class of differential equations, we refer to [Zw] and to Section ??.

As a final observation note that a number of the solution curves on Figure 1.10 appear to merge into one trajectory at certain regions of the display window. To see that this

MATLAB's Symbolic Toolbox which provides a link between the numerical powerhouse MATLAB and the symbolic computing engine Maple. The routine *dsolve* is certainly one of the most useful differential equation tools in the Symbolic Toolbox. For example, to find the solution of $y'(t) = y(t)^2 - t$ one simply types

$$\text{dsolve}('Dy = y^2 - t')$$

after the MATLAP prompt and pushes *Enter*.

is not the case one can use the zoom option in the dfield6 tool, or, one might use the crucial theoretical results of Section 1.5. As we will see there, under mild smoothness assumptions on the function $F(t, y)$, it is absolutely certain that the solution curves (trajectories) of an equation $y' = F(t, y)$ can never intersect.

Exercises

For each of the following differential equations, sketch a direction field on the rectangle $\mathcal{R} = \{(t, y) : -2 \le t, y \le 2\}$. You may do the direction fields by hand on graph paper using the points in $\mathcal{R}$ with integer coordinates as grid points. That is t and y are each chosen from the set $\{-2,\ -2,\ 0,\ 1,\ 2\}$. Alternatively, you may use a graphing calculator or a computer, where you could try 20 sample values for each of t and y, for a total of 400 grid points.

1. $y' = y - 1$

2. $y' = t$

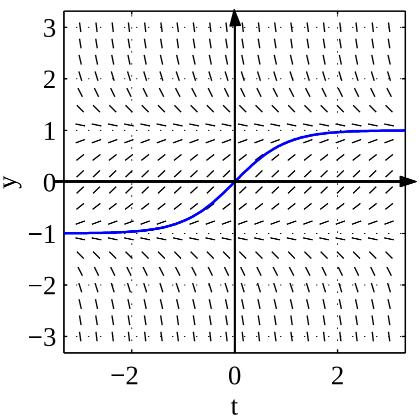
3. $y' = t^2$

4. $y' = y^2$

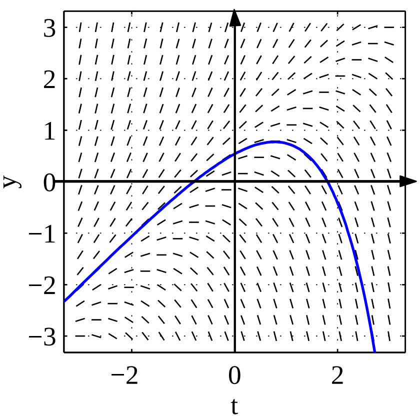
5. $y' = y(y + 1)$

   In Exercises 6 – 11, a differential equation is given together with its direction field. One solution is already drawn in. Draw at least five more representative solutions on the direction field. You may choose whatever initial conditions seem reasonable, or you can simply draw in the solutions with initial conditions $y(0) = -2,\ -1,\ 0,\ 1,$ and 2. Looking at the direction field can you tell if there are any constant solutions $y(t) = c$? If so, list them. Are there other straight line solutions that you can see from the direction field?

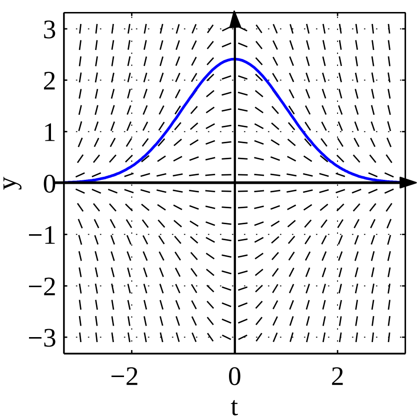
$y' = 1 - y^2$

6. 

$y' = y - t$

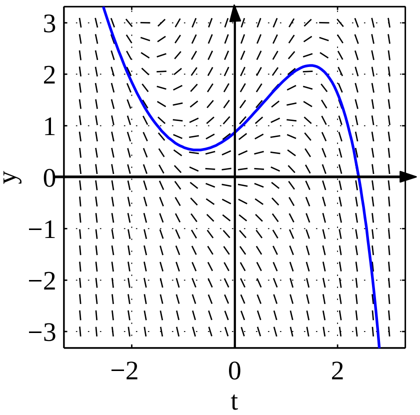
7. 

$y' = -ty$

8. 

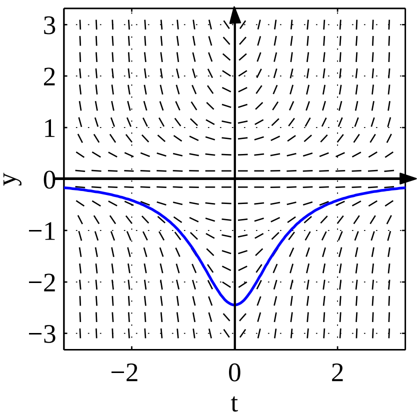
$$y' = y - t^2$$

9.



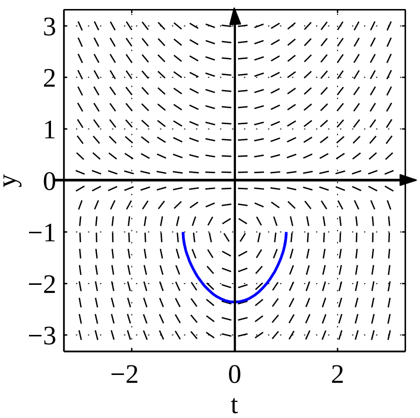
$$y' = ty^2$$

10.



$$y' = \frac{ty}{1+y}$$

11.



1.5 Existence and Uniqueness

Unfortunately, only a few simple types of differential equations can be solved explicitly in terms of well known elementary functions. In this section we will describe the method of **successive approximations**, which provides one of the many possible lines of attack for approximating solutions for arbitrary differential equations. This method, which is quite different from what most students have previously encountered, is the primary idea behind one of the main theoretical results concerning existence and uniqueness of solutions of the initial value problem

$$(*) \qquad y' = F(t,y), \quad y(t_0) = y_0,$$

where $F(t,y)$ is a *continuous* function of (t,y) in the rectangle

$$\mathcal{R} := \{(t,y) : a \le t \le b\,,\ c \le y \le d\}$$

and $(t_0, y_0) \in \mathcal{R}$. The key to the method of successive approximations is the fact that a continuously differentiable function $y(t)$ is a solution of $(*)$ if and only if it is a solution of the integral equation

$$(**) \qquad y(t) = y_0 + \int_{t_0}^{t} F(u, y(u))\,du.$$

To see the equivalence of the initial value problem $(*)$ and the integral equation $(**)$, we first integrate $(*)$ from t_0 to t and obtain $(**)$. Conversely, if $y(t)$ is a continuously differentiable solution of $(**)$, then $y(t_0) = y_0 + \int_{t_0}^{t_0} F(u, y(u))\,du = y_0$. Moreover, since $y(t)$ is a continuous function in t and $F(t,y)$ is a continuous function of (t,y), it follows that $g(t) := F(t, y(t))$ is a continuous function of t. Thus, by the Fundamental Theorem of Calculus,

$$y'(t) = \frac{d}{dt}\left(y_0 + \int_{t_0}^{t} F(u, y(u))\,du\right) = \frac{d}{dt}\left(y_0 + \int_{t_0}^{t} g(u)\,du\right) = g(t) = F(t, y(t)),$$

which is what it means to be a solution of $(*)$.

To solve the integral equation $(**)$, mathematicians have developed a variety of so-called "fixed point theorems", each of which leads to an existence and/or uniqueness result for solutions to the integral equation. One of the oldest and most widely used existence and uniqueness theorems is due to Émile Picard (1856-1941). Assuming that the function $F(t, y)$ is sufficiently "nice", he first employed the method of successive approximations to prove the existence and uniqueness of solutions of $(**)$. The method of successive approximations is an iterative procedure which begins with a crude approximation of a solution and improves it using a step by step procedure which brings us as close as we please to an exact and unique solution of $(**)$. The algorithmic procedure follows.

Algorithm 1.5.1 (Picard Approximation). Perform the following sequence of steps to produce an *approximate solution* to the integral equation $(**)$, and hence to initial value problem $(*)$.

(i) A rough initial approximation to a solution of $(**)$ is given by the constant function

$$y_0(t) := y_0.$$

(ii) Insert this initial approximation into the right hand side of equation $(**)$ and obtain the first approximation

$$y_1(t) := y_0 + \int_{t_0}^{t} F(u, y_0(u))\, du.$$

(iii) The next step is to generate the second approximation in the same way; i.e.,

$$y_2(t) := y_0 + \int_{t_0}^{t} F(u, y_1(u))\, du.$$

(iv) At the n-th stage of the process we have

$$y_n(t) := y_0 + \int_{t_0}^{t} F(u, y_{n-1}(u))\, du,$$

which is defined by substituting the previous approximation $y_{n-1}(t)$ into the right hand side of $(**)$.

It is one of Picard's great contributions to mathematics that he showed that the functions $y_n(t)$ converge to a unique, continuously differentiable solution $y(t)$ of $(**)$ (and thus of $(*)$) if the function $F(t, y)$ and its partial derivative $F_y(t, y) := \frac{\partial}{\partial y} F(t, y)$ are continuous functions of (t, y) on the rectangle $\mathcal{R}$.

Theorem 1.5.2 (Picard's Existence and Uniqueness Theorem). [4] *Let $F(t,y)$ and $F_y(t,y)$ be continuous functions of (t,y) on a rectangle*

$$\mathcal{R} = \{(t,y) : a \le t \le b\,,\ c \le y \le d\}\,.$$

If (t_0, y_0) is an interior point of $\mathcal{R}$, then there exists a unique solution $y(t)$ of

$$(*) \qquad y' = F(t,y)\,, \quad y(t_0) = y_0,$$

on some interval $[a', b']$ with $t_0 \in [a', b'] \subset [a, b]$. Moreover, the successive approximations $y_0(t) := y_0$,

$$y_n(t) := y_0 + \int_{t_0}^{t} F(u, y_{n-1}(u))\,du,$$

computed by Algorithm 1.5.1 converge towards $y(t)$ on the interval $[a', b']$. That is, for all $\epsilon > 0$ there exists n_0 such that the maximal distance between the graph of the functions $y_n(t)$ and the graph of $y(t)$ (for $t \in [a', b']$) is less than ϵ for all $n \ge n_0$.

If one only assumes that the function $F(t,y)$ is continuous on the rectangle $\mathcal{R}$, but makes no assumptions about $F_y(t,y)$, then Guiseppe Peano (1858-1932) showed that the initial value problem $(*)$ still has a solution on some interval I with $t_0 \in I \subset [a, b]$. This statement is known as **Peano's Existence Theorem**.[5] However, in this case the solutions are not necessarily unique (see Example 1.5.5 below). Theorem 1.5.2 is called a local existence and uniqueness theorem because it guarantees the existence of a unique solution in some interval $I \subset [a, b]$. In contrast, the following important variant of Picard's theorem yields a unique solution on the whole interval $[a, b]$.

Theorem 1.5.3. *Let $F(t,y)$ be a continuous function of (t,y) that satisfies a Lipschitz condition on a strip $\mathcal{S} = \{(t,y) : a \le t \le b\,,\ -\infty < y < \infty\}$. That is, assume that*

$$|F(t,y_1) - F(t,y_2)| \le K|y_1 - y_2|$$

for some constant $K > 0$. If (t_0, y_0) is an interior point of $\mathcal{S}$, then there exists a unique solution of

$$(*) \qquad y' = F(t,y)\,, \quad y(t_0) = y_0,$$

on the interval $[a, b]$.

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[4] A proof of this theorem can be found in G.F. Simmons' book *Differential Equations with Applications and Historical Notes*, 2nd edition McGraw-Hill, 1991.

[5] For a proof see, for example, A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis*, Chapter 3, Section 11, Dover 1975.

Example 1.5.4. Let us consider the Riccati equation $y' = y^2 - t$. Here, $F(t,y) = y^2 - t$ and $F_y(t,y) = 2y$ are continuous on all of $\mathbb{R}^2$. Thus, by Picard's Theorem 1.5.2, the initial value problem

$$(*) \qquad y' = y^2 - t\,, \quad y(0) = 0$$

has a unique solution on some (finite or infinite) interval I containing 0. The direction field for $y' = y^2 - t$ (see Section 1.4, Example 1.4.3) suggests that the maximal interval $I_{\max}$ on which the solution exists should be of the form $I_{\max} = (a, \infty)$ for some $-\infty \leq a < -1$. Observe that we can not apply Theorem 1.5.3 since

$$|F(t,y_1) - F(t,y_2)| = \left|(y_1^2 - t) - (y_2^2 - t)\right| = |y_1^2 - y_2^2| = |y_1 + y_2||y_1 - y_2|$$

can not be bounded by $K|y_1 - y_2|$ for some constant $K > 0$ because this would imply that $|y_1 + y_2| \leq K$ for all $-\infty < y_1, y_2 < \infty$. Thus, without further analysis of the problem, we have no precise knowledge about the maximal domain of the solution; i.e., we do not know if and where the solution will "blow up".

Next we show how Picard's method of successive approximations works in this example. To use this method we rewrite the initial value problem $(*)$ as an integral equation; i.e., we consider

$$(**) \qquad y(t) = \int_0^t (y(u)^2 - u)\, du.$$

We start with our initial approximation $y_0(t) = 0$, plug it into $(**)$ and obtain our first approximation

$$y_1(t) = \int_0^t (y_0(u)^2 - u)\, du = -\int_0^t u\, du = -\frac{1}{2}t^2.$$

The second iteration yields

$$y_2(t) = \int_0^t (y_1(u)^2 - u)\, du = \int_0^t \left(\frac{1}{4}u^4 - u\right)\, du = \frac{1}{4\cdot 5}t^5 - \frac{1}{2}t^2.$$

Since $y_2(0) = 0$ and

$$y_2(t)^2 - t = \frac{1}{4^2\cdot 5^2}t^{10} - \frac{1}{4\cdot 5}t^7 + \frac{1}{4}t^4 - t = \frac{1}{4^2\cdot 5^2}t^{10} - \frac{1}{4\cdot 5}t^7 + y_2'(t) \approx y_2'(t)$$

if t is close to 0, it follows that the second iterate $y_2(t)$ is already a "good" approximation of the exact solution for t close to 0. Since $y_2(t)^2 = \frac{1}{4^2\cdot 5^2}t^{10} - \frac{1}{4\cdot 5}t^7 + \frac{1}{4}t^4$, it follows that

$$y_3(t) = \int_0^t \left(\frac{1}{4^2\cdot 5^2}u^{10} - \frac{1}{4\cdot 5}u^7 + \frac{1}{4}u^4 - u\right)\, du = \frac{1}{11\cdot 4^2\cdot 5^2}t^{11} - \frac{1}{4\cdot 5\cdot 8}t^8 + \frac{1}{4\cdot 5}t^5 - \frac{1}{2}t^2.$$

According to Picard's theorem, the successive approximations $y_n(t)$ converge towards the exact solution $y(t)$, so we expect that $y_3(t)$ is an even better approximation of $y(t)$ for t close enough to 0.

Example 1.5.5. Consider the initial value problem

$$(*) \qquad y' = 3y^{2/3}\,, \quad y(t_0) = y_0.$$

The function $F(t, y) = y^{2/3}$ is continuous for all (t, y), so Peano's existence theorem shows that the initial value problem $(*)$ has a solution for all $-\infty < t_0, y_0 < \infty$. Moreover, since $F_y(t, y) = \dfrac{2}{y^{1/3}}$, Picard's existence and uniqueness theorem tells us that the solutions of $(*)$ are unique as long as the initial value $y_0 \neq 0$. Since the differential equation $y' = 3y^{2/3}$ is separable, we can rewrite it the differential form

$$\frac{1}{y^{2/3}}\,dy = 3dt,$$

and integrate the differential form to get

$$3y^{1/3} = 3t + c.$$

Thus, the functions $y(t) = (t+c)^3$ for $t \in \mathbb{R}$, together with the constant function $y(t) = 0$, are the solution curves for the differential equation $y' = 3y^{2/3}$, and $y(t) = (y_0^{1/3} + t - t_0)^3$ is the unique solution of the initial value problem $(*)$ if $y_0 \neq 0$. If $y_0 = 0$, then $(*)$ admits infinitely many solutions of the form

$$y(t) = \begin{cases} (t-\alpha)^3 & \text{if } t < \alpha \\ 0 & \text{if } \alpha \leq t \leq \beta \\ (t-\beta)^3 & \text{if } t > \beta, \end{cases} \tag{1}$$

where $t_0 \in [\alpha, \beta]$. The graph of one of these functions (where $\alpha = -1$, $\beta = 1$) is depicted in Figure 1.11. What changes among the different functions is the length of the straight line segment joining α to β on the t-axis.

Example 1.5.6. The differential equation

$$(\dagger) \qquad ty' = 3y$$

is separable (and linear). Thus, it is easy to see that $y(t) = ct^3$ is its general solution. In standard form Equation $(\dagger)$ is

$$(\ddagger) \qquad y' = \frac{3}{t}y$$

and the right hand side, $F(t, y) = \dfrac{3}{t}y$, is continuous provided $t \neq 0$. Thus Picard's theorem applies to give the conclusion that the initial value problem $y' = \dfrac{3}{t}y\,, \quad y(t_0) = y_0$

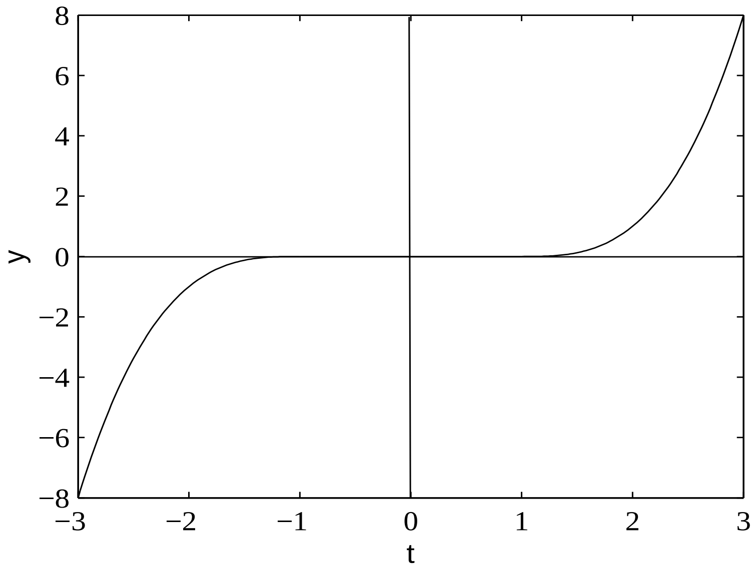


Figure 1.11: A solution (where $\alpha = -1$, $\beta = 1$ in Equation 1) of $y' = 3y^{2/3}$, $y(0) = 0$.

has a unique local solution if $t_0 \neq 0$ (given by $y(t) = \frac{y_0}{t_0^3} t^3$). However, if $t_0 = 0$, Picard's theorem contains no information about the existence and uniqueness of solutions. Indeed, in its standard form (‡), it is not meaningful to talk about solutions of this equation at $t = 0$ since $F(t, y) = \frac{3}{t} y$ is not even defined for $t = 0$. But in the originally designated form (†), where the t appears as multiplication on the left side of the equation, then an initial value problem starting at $t = 0$ makes sense, and moreover, the initial value problem

$$ty' = 3y\,, \quad y(0) = 0$$

has infinitely many solutions of the form $y(t) = ct^3$ for *any* $c \in \mathbb{R}$, whereas the initial value problem

$$ty' = 3y\,, \quad y(0) = y_0$$

has no solution if $y_0 \neq 0$. See Figure 1.12, where one can see that all of the function $y(t) = ct^3$ pass through the origin (i.e. $y(0) = 0$), but none pass through any other point on the y-axis.

Remark 1.5.7 (Geometric meaning of uniqueness).

1. The theorem on existence and uniqueness of solutions of differential equations (Theorem 1.5.2) has a particularly useful geometric interpretation. Suppose that $y' = F(t, y)$ is a first order differential equation for which Picard's existence and

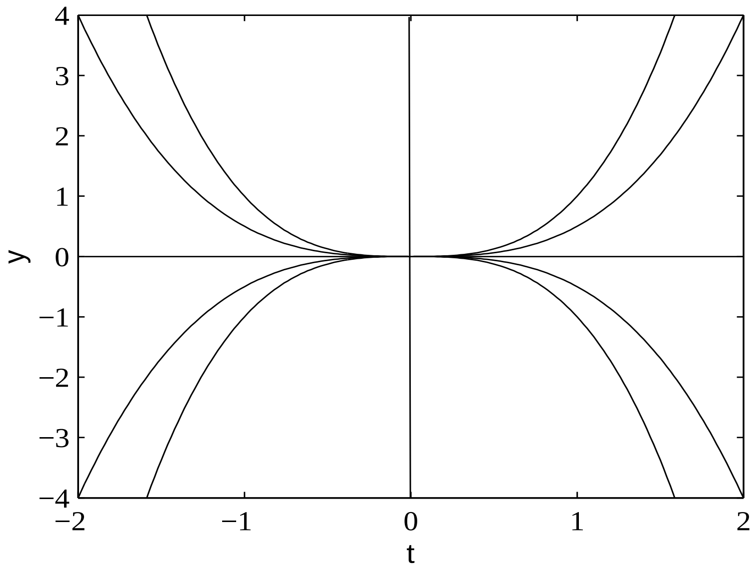


Figure 1.12: Distinct solutions of the initial value problem $ty' = 3y$, $y(0) = 0$.

uniqueness theorem applies. If $y_1(t)$ and $y_2(t)$ denote two different solutions of $y' = F(t, y)$, then the graphs of $y_1(t)$ and $y_2(t)$ can *never* intersect. The reason for this is just that if (t_0, y_0) is a point of the plane which is common to both the graph of $y_1(t)$ and that of $y_2(t)$, then **both** of these functions will satisfy the initial value problem

$$y' = F(t, y), \quad y(t_0) = y_0.$$

But if $y_1(t)$ and $y_2(t)$ are different functions, this will violate the uniqueness provision of Picard's theorem. Thus the situation depicted in Figures 1.11 and 1.12 where several solutions of the same differential equation go through the same point (in this case $(0, 0)$) can never occur for a differential equation which satisfies the hypotheses of Theorem 1.5.2. Similarly, the graphs of the function $y_1(t) = (t+1)^2$ and the constant function $y_2(t) = 1$ both pass through the point $(0, 1)$, and thus both cannot be solutions of the same differential equation satisfying Picard's theorem.

2. The above remark can be exploited in the following way. The constant function $y_1(t) = 0$ is a solution to the differential equation $y' = y^3 + y$ (check it). Since $F(t, y) = y^3 + y$ clearly has continuous partial derivatives, Picard's theorem applies. Hence, if $y_2(t)$ is a solution of the equation for which $y_2(0) = 1$, the above observation takes the form of stating that $y_2(t) > 0$ for *all* t. This is because, in order for $y(t)$ to ever be negative, it must first cross the t-axis, which is the graph of $y_1(t)$, and we have observed that two solutions of the same differential equation

can never cross. This observation will be further exploited in the next section.

Exercises

1. (a) Find the exact solution of the initial value problem

$$(*) \qquad y' = y^2, \quad y(0) = 1.$$

(b) Apply Picard's method (Theorem 1.5.2) to calculate the first three approximations $y_1(t)$, $y_2(t)$, and $y_3(t)$ to $(*)$ and compare these results with the exact solution.

► **Solution.** (a) The equation is separable so separate the variables to get $y^{-2}dy = dt$. Integrating gives $-y^{-1} = t + c$ and the initial condition $y(0) = 1$ implies that the integration constant $c = -1$, so that the exact solution of $(*)$ is

$$y(t) = \frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \cdots ; \quad |t| < 1.$$

(b) To apply Picard's method, let $y_0 = 1$ and define

$$\begin{aligned}
y_1(t) &= 1 + \int_0^t (y_0(s))^2\, ds = 1 + \int_0^t ds = 1 + t; \\
y_2(t) &= 1 + \int_0^t (y_1(s))^2\, ds = 1 + \int_0^t (1+s)^2\, ds = 1 + t + t^2 + \frac{t^3}{3}; \\
y_3(t) &= 1 + \int_0^t (y_2(s))^2\, ds = \int_0^t \left(1 + s + s^2 + \frac{s^3}{3}\right)^2 ds \\
&= 1 + \int_0^t \left(1 + 2s + 3s^2 + \frac{8}{3}s^3 + \frac{5}{3}s^4 + \frac{2}{3}s^5 + \frac{1}{9}s^6\right) ds \\
&= 1 + t + t^2 + t^3 + \frac{2}{3}t^4 + \frac{1}{3}t^5 + \frac{1}{9}t^6 + \frac{1}{63}t^7.
\end{aligned}$$

Comparing $y_3(t)$ to the exact solution, we see that the series agree up to order 3. ◄

2. Apply Picard's method to calculate the first three approximations $y_1(t)$, $y_2(t)$, $y_3(t)$ to the solution $y(t)$ of the initial value problem

$$y' = t - y, \quad y(0) = 1.$$

3. Apply Picard's method to calculate the first three approximations $y_1(t)$, $y_2(t)$, $y_3(t)$ to the solution $y(t)$ of the initial value problem

$$y' = t + y^2, \quad y(0) = 0.$$

Which of the following initial value problems are guaranteed a unique solution by Picard's theorem (Theorem 1.5.2)? Explain.

4. $y' = 1 + y^2, \quad y(0) = 0$

5. $y' = \sqrt{y}, \quad y(1) = 0$

6. $y' = \sqrt{y}, \quad y(0) = 1$

7. $y' = \dfrac{t-y}{t+y}, \quad y(0) = -1$

8. $y' = \dfrac{t-y}{t+y}, \quad y(1) = -1$

9. (a) Find the general solution of the differential equation

   (†) $$ty' = 2y - t.$$

   Sketch several specific solutions from this general solution.

   (b) Show that there is no solution to (†) satisfying the initial condition $y(0) = 2$. Why does this not contradict Theorem 1.5.2?

10. (a) Let t_0, y_0 be arbitrary and consider the initial value problem

    $$y' = y^2, \quad y(t_0) = y_0.$$

    Explain why Theorem 1.5.2 guarantees that this initial value problem has a solution on some interval $|t - t_0| \leq h$.

    (b) Since $F(t, y) = y^2$ and $F_y(t, y) = 2y$ are continuous on all of the $(t, y)-$plane, one might hope that the solutions are defined for all real numbers t. Show that this is not the case by finding a solution of $y' = y^2$ which is defined for all $t \in \mathbb{R}$ and another solution which is *not* defined for all $t \in \mathbb{R}$. (Hint: Find the solutions with $(t_0, y_0) = (0, 0)$ and $(0, 1)$.)

11. Is it possible to find a function $F(t, y)$ that is continuous and has a continuous partial derivative $F_y(t, y)$ such that the two functions $y_1(t) = t$ and $y_2(t) = t^2 - 2t$ are both solutions to $y' = F(t, y)$ on an interval containing 0?

12. Show that the function

    $$y_1(t) = \begin{cases} 0, & \text{for } t < 0 \\ t^3 & \text{for } t \geq 0 \end{cases}$$

    is a solution of the initial value problem $ty' = 3y$, $y(0) = 0$. Show that $y_2(t) = 0$ for all t is a second solution. Explain why this does not contradict Theorem 1.5.2.

1.6 Miscellaneous Nonlinear First Order Equations

We have learned how to find explicit solutions for the standard first order differential equation

$$y' = F(t, y)$$

when the right hand side of the equation has one of the particularly simple forms:

1. $F(t, y) = h(t)g(y)$, i.e., the equation is *separable*, or
2. $F(t, y) = -p(t)y + f(t)$, i.e., the equation is *linear*.

Unfortunately, in contrast to the separable and first order linear differential equations, for an arbitrary function $F(t, y)$ it is very difficult to find closed form "solution formulas". In fact, most differential equations do not have closed form solutions and one has to resort to numerical or asymptotic approximation methods to gain information about them. In this section we discuss some other types of first-order equations which you may run across in applications and that allow closed form solutions in the same sense as the separable and first order linear differential equations. That is, the "explicit" solution may very well involve the computation of an indefinite integral which cannot be expressed in terms of elementary functions, or the solution may be given implicitly by an equation which cannot be reasonably solved in terms of elementary function. Our main purpose in this section is to demonstrate techniques that allow us to find solutions of these types of first-order differential equations and we completely disregard in this section questions of continuity, differentiability, vanishing divisors, and so on. If you are interested in the huge literature covering other special types of first order differential equations for which closed form solutions can be found, we refer you to books like [Zw] or to one of the three M's (Mathematica, Maple, or MatLab) which are, most likely, more efficient in computing closed form solutions than most of us will ever be.

Exact Differential Equations

A particularly important class of nonlinear first order differential equations that can be solved (explicitly or implicitly) is that of **exact first order equations**. To explain the mathematics behind exact equations, it is necessary to recall some facts about calculus of functions of two variables.[6] Let $V(t, y)$ be a function of two variables defined on a rectangle

$$\mathcal{R} := \{(t, y) : a \le t \le b\,,\ c \le y \le d\}\,.$$

[6]The facts needed will be found in any calculus textbook. For example, you may consult Chapter 14 of *Calculus: Early Transcendentals, Fourth Edition* by James Stewart, Brooks-Cole, 1999.

The curve with equation $V(t, y) = c$, where $c \in \mathbb{R}$ is a constant, is a **level curve** of V.

Example 1.6.1.

1. If $V(t, y) = t + 2y$, the level curves are all of the lines $t + 2y = c$ of slope -0.5.

2. If $V(t, y) = t^2 + y^2$, the level curves are the circles $t^2 + y^2 = c$ centered at $(0, 0)$ of radius $\sqrt{c}$, provided $c > 0$. If $c = 0$, then the level "curve" $t^2 + y^2 = 0$ consists of the single point $(0, 0)$, while if $c < 0$ there are no points at all which solve the equation $t^2 + y^2 = c$.

3. If $V(t, y) = t^2 - y^2$ then the level curves of V are the hyperbolas $t^2 - y^2 = c$ if $c \neq 0$, while the level curve $t^2 - y^2 = 0$ consists of the two lines $y = \pm t$.

4. If $V(t, y) = y^2 - t$ then the level curves are the parabolas $y^2 - t = c$ with axis of symmetry the t-axis and opening to the right.

Thus we see that sometimes a level curve defines y explicitly as a function of t (for example, $y = \frac{1}{2}(c - t)$ in number 1 above), sometimes t is defined explicitly as a function of y (for example, $t = -2y + c$ in number 1, and $t = y^2 - c$ in number 3 above), while in other cases it may only be possible to define y as a function of t (or t as a function of y) implicitly by the level curve equation $V(t, y) = c$. For instance, the level curve $t^2 + y^2 = c$ for $c > 0$ defines y as a function of t in two ways ($y = \pm\sqrt{c - t^2}$ for $-\sqrt{c} < t < \sqrt{c}$) and it also defines t as a function of y in two ways ($t = \pm\sqrt{c - y^2}$ for $-\sqrt{c} < y < \sqrt{c}$).

If we are given a two variable function $V(t, y)$ is there anything which can be said about *all* of the level curves $V(t, y) = c$? The answer is yes. What the level curves of a fixed two variable function have in common is that *every one of the functions $y(t)$ defined implicitly by $V(t, y) = c$, no matter what c is, is a solution of the same differential equation.* The mathematics underlying this observation is the chain rule in two variables, which implies that

$$\frac{d}{dt} V(t, y(t)) = V_t(t, y(t)) + V_y(t, y(t))y'(t),$$

where V_t, V_y denote the partial derivatives of $V(t, y)$ with respect to t and y, respectively. Thus, if a function $y(t)$ is given implicitly by a level curve

$$V(t, y(t)) = c,$$

then $y(t)$ satisfies the equation

$$0 = \frac{d}{dt} c = \frac{d}{dt} V(t, y(t)) = V_t(t, y(t)) + V_y(t, y(t))y'(t).$$

This means that $y(t)$ is a solution of the differential equation

$$V_t(t, y) + V_y(t, y)y' = 0. \tag{1}$$

Notice that the constant c does not appear anywhere in this equation so that *every* function $y(t)$ determined implicitly by a level curve of $V(t, y)$ satisfies this same equation. An equation of the form given by Equation 1 is referred to as an exact equation:

Definition 1.6.2. A differential equation written in the form

$$M(t, y) + N(t, y)y' = 0$$

is said to be **exact** if there is a function $V(t, y)$ such that $M(t, y) = V_t(t, y)$ and $N(t, y) = V_y(t, y)$.

What we observed above is that, if $y(t)$ is defined implicitly by a level curve $V(t, y) = c$, then $y(t)$ is a solution of the exact equation 1. Moreover, *the level curves determine all of the solutions* of Equation 1, so the general solution is defined by

$$\boxed{V(t, y) = c.} \tag{2}$$

Example 1.6.3.

1. The exact differential equation determined by $V(t, y) = t + 2y$ is

$$0 = V_t(t, y) + V_y(t, y)y' = 1 + 2y'$$

so the general solution of $1 + 2y' = 0$ is $t + 2y = c$.

2. The exact differential equation determined by $V(t, y) = t^2 + y^2$ is

$$0 = V_t(t, y) + V_y(t, y)y' = 2t + 2yy'.$$

Hence, the general solution of the equation $t + yy' = 0$, which can be written in standard form as $y' = -t/y$, is $t^2 + y^2 = c$.

Suppose we are given a differential equation in the form

$$M(t, y) + N(t, y)y' = 0,$$

but we are not given apriori that $M(t, y) = V_t(t, y)$ and $N(t, y) = V_y(t, y)$. How can we determine if there is such a function $V(t, y)$, and if there is, how can we find it? That is, is there a *criterion* for determining if a given differential equation is exact, and if so is there a *procedure* for producing the function $V(t, y)$ whose level curves implicitly determine the solutions. The answer to both questions is yes. The criterion for exactness is given by the following theorem; the procedure for finding $V(t, y)$ will be illustrated by example.

Theorem 1.6.4 (Criterion for exactness). *A first order differential equation*

$$M(t, y) + N(t, y)y' = 0$$

in which $M(t, y)$ and $N(t, y)$ have continuous first order partial derivatives is exact if and only if

$$\boxed{M_y(t, y) = N_t(t, y)} \tag{3}$$

for all t, y in a square region of $\mathbb{R}^2$.

Proof. Recall (from your calculus course) that all functions $V(t, y)$ whose second partial derivatives exist and are continuous satisfy

$$(*) \qquad V_{ty}(t, y) = V_{yt}(t, y),$$

where $V_{ty}(t, y)$ denotes the derivative of $V_t(t, y)$ with respect to y, and $V_{yt}(t, y)$ is the derivative of $V_y(t, y)$ with respect to t. The equation $(*)$ is known as Clairaut's theorem (after Alexis Clairaut (1713 – 1765)) on the equality of mixed partial derivatives. If the equation $M(t, y) + N(t, y)y' = 0$ is exact then (by definition) there is a function $V(t, y)$ such that $V_t(t, y) = M(t, y)$ and $V_y(t, y) = N(t, y)$. Then by Clairaut's theorem,

$$M_y(t, y) = \frac{\partial}{\partial y} V_t(t, y) = V_{ty}(t, y) = V_{yt}(t, y) = \frac{\partial}{\partial t} V_y(t, y) = N_t(t, y).$$

Hence condition 3 is satisfied.

Now assume, conversely, that condition 3 is satisfied. To verify that the equation $M(t, y) + N(t, y)y' = 0$ is exact, we need to search for a function $V(t, y)$ which satisfies the equations

$$V_t(t, y) = M(t, y) \text{ and } V_y(t, y) = N(t, y).$$

The procedure will be sketched and then illustrated by means of an example. The equation $V_t(t, y) = M(t, y)$ means that we should be able to recover $V(t, y)$ from $M(t, y)$ by indefinite integration:

$$V(t, y) = \int M(t, y)\, dt + \varphi(y). \tag{4}$$

The function $\varphi(y)$ appears as the "integration constant" since any function of y goes to 0 when differentiated with respect to t. The function $\varphi(y)$ can be determined from the equation

$$V_y(t, y) = \frac{\partial}{\partial y} \int M(t, y)\, dt + \varphi'(y) = N(t, y). \tag{5}$$

That is

$$\varphi'(y) = N(t, y) - \frac{\partial}{\partial y} \int M(t, y)\, dt. \tag{6}$$

The verification that the function on the right is really a function only of y (as it must be if it is to be $\varphi'(y)$) is where condition 3 is needed. □

Example 1.6.5. Solve the differential equation $y' = \dfrac{t - y}{t + y}$

► **Solution.** We rewrite the equation in the form $y - t + (t + y)y' = 0$ to get that $M(t, y) = y - t$ and $N(t, y) = y + t$. Since $M_y(t, y) = 1 = N_t(t, y)$, it follows that the equation is exact and the general solution will have the form $V(t, y) = c$, where $V_t(t, y) = y - t$ and $V_y(t, y) = y + t$. Since $V_t(t, y) = y - t$ it follows that

$$V(t, y) = \int (y - t)\, dt + \varphi(y) = yt - \frac{t^2}{2} + \varphi(y),$$

where $\varphi(y)$ is a yet to be determined function depending on y, but not on t. To determine $\varphi(y)$ note that $y + t = V_y(t, y) = t + \varphi'(y)$, so that $\varphi'(y) = y$. Hence

$$\varphi(y) = \frac{y^2}{2} + c_1$$

for some arbitrary constant c_1, and thus

$$V(t, y) = yt - \frac{t^2}{2} + \frac{y^2}{2} + c_1.$$

The general solution of $y' = \dfrac{t - y}{t + y}$ is therefore given by the implicit equation

$$V(t, y) = yt - \frac{t^2}{2} + \frac{y^2}{2} + c_1 = c.$$

This is the form of the solution which we are led to by our general solution procedure outlined in the proof of Theorem 1.6.4. However, after further simplifying this equation and renaming constants several times the general solution can be expressed implicitly by

$$2yt - t^2 + y^2 = c,$$

and explicitly by

$$y(t) = -t \pm \sqrt{2t^2 + c}.$$

◄

What happens if we try to solve by equation $M(t,y)+N(t,y)y' = 0$ by the procedure outlined above without *first* verifying that it is exact? If the equation is not exact, you will discover this fact when you get to Equation 6, since $\varphi'(y)$ will *not* be a function only of y, as the following example illustrates.

Example 1.6.6. Try to solve the equation $(t - 3y) + (2t + y)y' = 0$ by the solution procedure for exact equations.

▶ **Solution.** Note that $M(t,y) = t - 3y$ and $N(t,y) = 2t + y$. First apply Equation 4 to get

$$(\dagger) \qquad V(t,y) = \int M(t,y), dt = \int (t - 3y)\,dt = \frac{t^2}{2} - 3ty + \varphi(y),$$

and then determine $\varphi(y)$ from Equation 6:

$$(\ddagger) \qquad \varphi'(y) = N(t,y) - \frac{\partial}{\partial y}\int M(t,y)\,dt = (2t+y) - \frac{\partial}{\partial y}\left(\frac{t^2}{2} - 3ty + \varphi(y)\right) = y - t.$$

But we see that there is a problem since $\varphi'(y)$ in (‡) involves both y and t. This is where it becomes obvious that you are not dealing with an exact equation, and you cannot proceed with this procedure. Indeed, $M_y(t,y) = -3 \neq 2 = N_t(t,y)$, so that this equation fails the exactness criterion 3. ◀

Bernoulli Equations

It is sometimes possible to change the variables in a differential equation $y' = F(t,y)$ so that in the new variables the equation appears in a form you already know how to solve. This is reminiscent of the substitution procedure for computing integrals. We will illustrate the procedure with a class of equations known as **Bernoulli equations** (named after Jakoub Bernoulli, (1654 – 1705)), which are equations of the form

$$y' + p(t)y = f(t)y^n. \qquad (7)$$

If $n = 0$ this equation is linear, while if $n = 1$ the equation is both separable and linear. Thus, it is the cases $n \neq 0,\ 1$ where a new technique is needed. Start by dividing Equation 7 by y^n to get

$$(*) \qquad y^{-n}y' + p(t)y^{1-n} = f(t),$$

and notice that if we introduce a new variable $z = y^{1-n}$, then the chain rule gives

$$z' = \frac{dz}{dt} = \frac{dz}{dy}\frac{dy}{dt} = (1-n)y^{-n}y',$$

and Equation (∗), after multiplying by the constant $(1-n)$, becomes a *linear* first order differential equation in the variables t, z:

$$(**) \qquad z' + (1-n)p(t)z = f(t).$$

Equation (∗∗) can then be solved by Algorithm 1.3.5, and the solution to 7 is obtained by solving $z = y^{1-n}$ for y.

Example 1.6.7. Solve the Bernoulli equation $y' + y = y^2$.

► **Solution.** In this equation $n = 2$, so if we let $z = y^{1-2} = y^{-1}$, we get $z' = -y^{-2}y'$. After dividing our equation by y^2 we get $y^{-2}y' + y^{-1} = 1$, which in terms of the variable z is $-z' + z = 1$. In the standard form for linear equations this becomes

$$z' - z = -1.$$

We can apply Algorithm 1.3.5 to this equation. The integrating factor will be e^-t. Multiplying by the integrating factor gives $(e^{-1}z)' = -e^{-t}$ so that $e^{-t}z = e^{-t} + c$. Hence $z = 1 + ce^t$. Now go back to the original function y by solving $z = y^{-1}$ for y. Thus

$$y = z^{-1} = (1 + ce^t)^{-1} = \frac{1}{1 + ce^t}$$

is the general solution of the Bernoulli equation $y' + y = y^2$.

Note that this equation is also a separable equation, so it could have been solved by the technique for separable equations, but the integration (and subsequent algebra) involved in the current procedure is simpler. ◄

There are a number of other types of substitutions which are used to transform certain differential equations into a form which is more amenable for solution. We will not pursue the topic further in this text. See the book [Zw] for a collection of many different solution algorithms.

Exercises

Exact Equations

For Exercises 1 – 9, determine if the equation is exact, and if it is exact, find the general solution.

1. $(y^2 + 2t) + 2tyy' = 0$

▶ **Solution.** This can be written in the form $M(t,y) + N(t,y)y' = 0$ where $M(t,y) = y^2 + 2t$ and $N(t,y) = 2ty$. Since $M_y(t,y) = 2y = N_t(t,y)$, the equation is **exact** (see Equation (3.2.2)), and the general solution is given implicitly by $F(t,y) = c$ where the function $F(t,y)$ is determined by $F_t(t,y) = M(t,y) = y^2 + 2t$ and $F_y(t,y) = N(t,y) = 2ty$. These equations imply that $F(t,y) = t^2 + ty^2$ will work so the solutions are given implicitly by $t^2 + ty^2 = c$. ◀

2. $y - t + ty' + 2yy' = 0$

3. $2t^2 - y + (t + y^2)y' = 0$

4. $y^2 + 2tyy' + 3t^2 = 0$

5. $(3y - 5t) + 2yy' - ty' = 0$

6. $2ty + (t^2 + 3y^2)y' = 0$, $y(1) = 1$

7. $2ty + 2t^2 + (t^2 - y)y' = 0$

8. $t^2 - y - ty' = 0$

9. $(y^3 - t)y' = y$

10. Find conditions on the constants a, b, c, d which guarantee that the differential equation $(at + by) = (ct + dy)y'$ is exact.

    Bernoulli Equations. Find the general solution of each of the following Bernoulli equations. If an initial value is given, also solve the initial value problem.

11. $y' - y = ty^2, \quad y(0) = 1$

12. $y' + ty = t^3y^3$

13. $(1 - t^2)y' - ty = 5ty^2$

14. $y' + ty = ty^3$

15. $y' + y = ty^3$

    General Equations. The following problems may any of the types studied so far.

16. $y' = ty - t, \quad y(1) = 2$

17. $(t^2 + 3y^2)y' = -2ty$

18. $t(t+1)y' = 2\sqrt{y}$

19. $y' = \dfrac{y}{t^2 + 2t - 3}$

20. $\sin y + y\cos t + 2t + (t\cos y + \sin t)y' = 0$

21. $y' + \dfrac{1}{t(t-1)}y = t - 1$

22. $y' - y = \frac{1}{2}e^t y^{-1}, \quad y(0) = -1$

23. $y' = \dfrac{8t^2 - 2y}{t}$

24. $y' = \dfrac{y^2}{t}, \quad y(1) = 1$

Chapter 2

THE LAPLACE TRANSFORM

In this chapter we introduce the **Laplace Transform** and show how it gives a direct method for solving certain initial value problems. This technique is extremely important in applications since it gives an easily codified procedure that goes directly to the solution of an initial value problem without first determining the general solution of the differential equation. The same theoretical procedure applies to ordinary differential equations of arbitrary order (with constant coefficients) and even to systems of constant coefficient linear ordinary differential equations, which will be treated in Chapter 6. Moreover the same procedure applies to linear constant coefficient equations (of any order) for which the forcing function is not necessarily continuous. This will be addressed in Chapter 4.

You are already familiar with certain operators which transform one function into another. One particularly important example is the differentiation operator $\mathbf{D}$ which transforms each function which has a derivative into its derivative, i.e., $\mathbf{D}(f) = f'$. The Laplace transform $\mathcal{L}$ is an integral operator on certain spaces of functions on the interval $[0, \infty)$. By an **integral operator**, we mean an operator T which takes an input function f and transforms it into another function $F = T\{f\}$ by means of integration with a **kernel** function $K(s, t)$. That is,

$$T\{f(t)\} = \int_0^\infty K(s, t) f(t)\, dt = F(s).$$

The Laplace transform is the particular integral transform obtained by using the kernel function

$$K(s, t) = e^{-st}.$$

When applied to a (constant coefficient linear) differential equation the Laplace transform turns it into an algebraic equation, one that is generally much easier to solve. After solving the algebraic equation one needs to transform the solution of the algebraic equation back into a function that is the solution to the original differential equation. This last step is known as the **inversion problem**.

This process of transformation and inversion is analogous to the use of the logarithm to solve a multiplication problem. When scientific and engineering calculations were done by hand, the standard procedure for doing multiplication was to use logarithm tables to turn the multiplication problem into an addition problem. Addition, by hand, is much easier than multiplication. After performing the addition, the log tables were used again, in reverse order, to complete the calculation. Now that calculators are universally available, multiplication is no more difficult than addition (one button is as easy to push as another) and the use of log tables as a tool for multiplication is essentially extinct. The same cannot be said for the use of Laplace transforms as a tool for solving ordinary differential equations. The use of sophisticated mathematical software (Maple, Mathematica, MatLab) can simplify many of the routine calculations necessary to apply the Laplace transform, but it in no way absolves us of the necessity of having a firm theoretical understanding of the underlying mathematics, so that we can legitimately interpret the numbers and pictures provided by the computer. For the purposes of this course, we provide a table (Table C.2) of Laplace transforms for many of the common functions you are likely to see. This will provide a basis for studying many examples.

2.1 Definition of The Laplace Transform

If $f(t)$ is a function defined for all $t \geq 0$, then the **Laplace transform** of f is the function $\mathcal{L}\{f(t)\}(s) = F(s)$ defined by the equation

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{r\to\infty} \int_0^r e^{-st} f(t)\,dt \tag{1}$$

provided the limit exists for all sufficiently large s. This means that there is a number N, which will depend on the function f, so that the limit exists whenever $s > N$. If there is no such N, then the function f will not have a Laplace transform.

Let's analyze this equation somewhat further. The function f with which we start will sometimes be called the **input function**. Generally, 't' will denote the variable for an input function f, while the Laplace transform of f, denoted $\mathcal{L}\{f\}(s)$, is a new function (the **output function**), whose variable will usually be 's'. Thus Equation (1) is a formula for computing the value of the function $\mathcal{L}\{f\}$ at the particular point s, so that, in particular

$$F(2) = \mathcal{L}\{f\}(2) = \int_0^\infty e^{-2t} f(t)\,dt \quad \text{and} \quad F(-3) = \mathcal{L}\{f\}(-3) = \int_0^\infty e^{3t} f(t)\,dt,$$

provided $s = 2$ and $s = -3$ are in the domain of $\mathcal{L}\{f\}$.

Normally, we will use a lower case letter to denote the input function and the corresponding uppercase letter to denote its Laplace transform. Thus, $F(s)$ is the Laplace transform of $f(t)$, $Y(s)$ is the Laplace transform of $y(t)$, etc. Hence there are two distinct notations that we will be using for the Laplace transform. Thus

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{and} \quad \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

are interchangeable notations for the same function of s. It is also worth emphasizing that, while the input function f has a well determined domain $[0, \infty)$, the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ is only defined for all sufficiently large s, and the domain will depend on the particular input function f. In practice this will not be a problem, and we will generally not emphasize the particular domain of $F(s)$.

In this chapter we will only consider continuous input functions. However, later we will ease this restriction and consider Laplace transforms of some functions which are not continuous.

A particularly useful property of the Laplace transform, both theoretically and computationally, is that of **linearity**. For the Laplace transform linearity means the following, which, because of its importance, we state formally as a theorem.

Theorem 2.1.1. *The Laplace transform is linear. In other words, if f and g are input functions and a and b are constants then*

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}.$$

Proof. This follows from the fact that (improper) integration is linear. □

Laplace Transform of Elementary Functions

Example 2.1.2 (Constant Functions). Compute the Laplace transform of the constant function 1.

► **Solution.** For the constant function 1 we have

$$\begin{aligned}\mathcal{L}\{1\}(s) &= \int_0^\infty e^{-st}\cdot 1\,dt = \lim_{r\to\infty} \left.\frac{e^{-ts}}{-s}\right|_0^r \\ &= \lim_{r\to\infty} \frac{e^{-rs}-1}{-s} = \frac{1}{s} \qquad \text{for } s > 0.\end{aligned}$$

◄

Some comments are in order. The condition $s > 0$ is needed for the limit

$$\lim_{r\to\infty} \frac{e^{-rs} - 1}{-s}$$

that defines the improper integral $\int_0^\infty e^{-st}\,dt$ to exist. This is because

$$\lim_{r\to\infty} e^{rc} = \begin{cases} 0 & \text{if } c < 0 \\ \infty & \text{if } c > 0. \end{cases}$$

More generally, it follows from L'Hôpital's rule that

$$\lim_{t\to\infty} t^n e^{ct} = 0 \quad \text{if } n \geq 0 \text{ and } c < 0. \tag{2}$$

This important fact (which you learned in calculus) is used in a number of calculations in the following manner. We will use the notation $h(t)|_a^\infty$ as a shorthand for $\lim_{r\to\infty} h(t)|_a^r = \lim_{r\to\infty}(h(r) - h(a))$. In particular, if $\lim_{t\to\infty} h(t) = 0$ then $h(t)|_a^\infty = -h(a)$, so that Equation (2) implies

$$t^n e^{ct}\Big|_0^\infty = \begin{cases} 0 & \text{if } n > 0 \text{ and } c < 0 \\ -1 & \text{if } n = 0 \text{ and } c < 0. \end{cases} \tag{3}$$

Example 2.1.3 (Power functions). Compute the Laplace transform of t^n.

► **Solution.** If $n = 0$ then $f(t) = t^0 = 1$ and this case is thus given above. Assume now that $n > 0$. Then

$$\mathcal{L}\{t^n\}(s) = \int_0^\infty e^{-st} t^n\,dt$$

and this integral can be computed using integration by parts with the choice of u and dv from the following table:

$u = t^n$	$dv = e^{-st}\,dt$
$du = nt^{n-1}\,dt$	$v = \dfrac{-e^{-st}}{s}$

Using this table and the observations concerning L'Hôpital's rule in the previous paragraph, we find that if $n > 0$ and $s > 0$, then

$$\begin{aligned} \mathcal{L}\{t^n\}(s) &= \int_0^\infty e^{-st} t^n\,dt \\ &= t^n \frac{e^{-st}}{-s}\Big|_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\,dt \\ &= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}(s). \end{aligned}$$

By iteration of this process (or by induction), we obtain (again assuming $n > 0$ and $s > 0$)

$$\begin{aligned}
\mathcal{L}\{t^n\}(s) &= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}(s) \\
&= \frac{n}{s} \cdot \frac{(n-1)}{s}\mathcal{L}\left\{t^{n-2}\right\}(s) \\
&= \frac{n}{s} \cdot \frac{n-1}{s} \cdots \frac{2}{s} \cdot \frac{1}{s}\mathcal{L}\left\{t^0\right\}(s).
\end{aligned}$$

But $\mathcal{L}\{t^0\}(s) = \mathcal{L}\{1\}(s) = 1/s$ so we conclude

$$\boxed{\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}, \qquad s > 0.}$$

◀

Example 2.1.4 (The exponential function). Compute the Laplace transform of e^{at}

▶ **Solution.**

$$\mathcal{L}\left\{e^{at}\right\}(s) = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty .$$

From Equation (3), the right hand limit evaluates to $1/(s-a)$ provided the coefficient of t in the exponential is negative. That is, provided $s > a$. Hence,

$$\boxed{\mathcal{L}\left\{e^{at}\right\}(s) = \frac{1}{s-a}, \qquad s > a.}$$

◀

We note that in this example the calculation can be justified for $a \in \mathbb{C}$, once we have noted what we mean by the complex exponential function. The main thing that we want to note is that the complex exponential function e^z ($z \in \mathbb{C}$) satisfies the same rules of algebra as the real exponential function, namely, $e^{z_1+z_2} = e^{z_1}e^{z_2}$. This is achieved by simply noting that the same power series which defines the real exponential makes sense for complex values also. Recall that the exponential function e^x has a power series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which converges for all $x \in \mathbb{R}$. This infinite series makes perfectly good sense if x is replaced by *any* complex number z, and moreover, it can be shown that the resulting series converges for all $z \in \mathbb{C}$. Thus, we *define* the **complex exponential function** by means of the convergent series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{4}$$

It can be shown that this function e^z satisfies the expected functional equation, that is

$$e^{z_1+z_2} = e^{z_1} e^{z_2}.$$

Since $e^0 = 1$, it follows that $\dfrac{1}{e^z} = e^{-z}$. Taking $z = it$ in Definition 4 leads to an important formula for the real and imaginary parts of e^{it}:

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \cdots \\ &= (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots) + i(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots) = \cos t + i \sin t, \end{aligned}$$

where one has to know (from studying calculus) that the two series following the last equality are the Taylor series expansions for $\cos t$ and $\sin t$, respectively. In words, this says that the real part of e^{it} is $\cos t$ and the imaginary part of e^{it} is $\sin t$. Combining this with the basic exponential functional property gives the formula, known as **Euler's formula**, for the real and imaginary parts of $e^{\alpha t}$ ($\alpha = a + bi$):

$$e^{\alpha t} = e^{(a+bi)t} = e^{at+ibt} = e^{at} e^{ibt} = e^{at}(\cos bt + i \sin bt).$$

We formally state this as a theorem.

Theorem 2.1.5 (Euler's Formula). *If $\alpha = a + bi \in \mathbb{C}$ and $t \in \mathbb{R}$, then*

$$e^{\alpha t} = e^{at}(\cos bt + i \sin bt). \tag{5}$$

An important conclusion of Euler's formula is the limit formula

$$\lim_{t\to\infty} e^{(a+bi)t} = 0, \quad \text{if } a < 0.$$

More generally, the analog of Equation (3) (which also follows from Equation (2)) is

$$t^n e^{(a+bi)t}\Big|_0^\infty = \begin{cases} 0 & \text{if } n > 0 \text{ and } a < 0 \\ -1 & \text{if } n = 0 \text{ and } a < 0. \end{cases} \tag{6}$$

Example 2.1.6 (The complex exponential function). Compute the Laplace transform of $e^{\alpha t}$, where $\alpha = a + bi$.

► **Solution.**

$$\begin{aligned}\mathcal{L}\left\{e^{\alpha t}\right\}(s) &= \int_0^\infty e^{-st}e^{\alpha t}\,dt \\ &= \int_0^\infty e^{-(s-\alpha)t}\,dt = \left.\frac{e^{-(s-\alpha)t}}{-(s-\alpha)}\right|_0^\infty.\end{aligned}$$

From Equation (6), the right hand limit evaluates to $1/(s-\alpha)$ provided the *real part of the coefficient of* t in the exponential, i.e., $-(s-a)$, is negative. That is, provided $s > a$. Hence,

$$\boxed{\mathcal{L}\left\{e^{\alpha t}\right\}(s) = \frac{1}{s-\alpha}, \qquad s > a = \operatorname{Re}\alpha.}$$

◄

Example 2.1.7 (Sine and Cosine). Compute the Laplace transform of $\sin at$ and $\cos at$.

► **Solution.** A direct application of the definition of the Laplace Transform applied to sin or cos would each require two integrations by parts; a tedious calculation. Linearity and the use of the complex exponential function simplifies this substantially. On the one hand, we have

$$\begin{aligned}\mathcal{L}\left\{e^{ibt}\right\}(s) &= \frac{1}{s-ib} \\ &= \frac{1}{s-ib}\,\frac{s+ib}{s+ib} = \frac{s+ib}{s^2+b^2} \\ &= \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{L}\left\{e^{ibt}\right\}(s) &= \mathcal{L}\left\{\cos bt + i\sin bt\right\}(s) \quad \text{by Euler's Formula} \\ &= \mathcal{L}\left\{\cos bt\right\}(s) + i\mathcal{L}\left\{\sin bt\right\}(s) \quad \text{by linearity.}\end{aligned}$$

By equating the real and imaginary parts we obtain

$$\mathcal{L}\left\{\cos bt\right\}(s) = \frac{s}{s^2+b^2} \quad \text{and} \quad \mathcal{L}\left\{\sin bt\right\}(s) = \frac{b}{s^2+b^2}$$

◄

Example 2.1.8. Compute the Laplace transform of $t^n e^{at}$.

► **Solution.** Notice that

$$\mathcal{L}\left\{t^n e^{at}\right\}(s) = \int_0^\infty e^{-st} t^n e^{at}\, dt = \int_0^\infty e^{-(s-a)} t^n\, dt = \mathcal{L}\left\{t^n\right\}(s-a). \tag{7}$$

What this formula says is that the Laplace transform of the function $t^n e^{at}$ evaluated at the point s is the same as the Laplace transform of the function t^n evaluated at the point $s - a$. Since $\mathcal{L}\{t^n\}(s) = n!/s^{n+1}$, we conclude

$$\boxed{\mathcal{L}\left\{t^n e^{at}\right\}(s) = \frac{n!}{(s-a)^{n+1}}, \quad \text{for } s > a.} \tag{8}$$

We note that this formula is also valid for $a \in \mathbb{C}$, where the condition $s > a$ will be replaced by $s > \operatorname{Re} a$. ◄

As special cases of this example, we note that

$$\mathcal{L}\left\{te^{2t}\right\} = \frac{1}{(s-2)^2}, \quad \mathcal{L}\left\{t^2 e^{t}\right\} = \frac{2}{(s-1)^3}, \quad \text{and} \quad \mathcal{L}\left\{t^3 e^{-2t}\right\} = \frac{6}{(s+2)^4}.$$

If the function t^n in Equation (7) is replaced by an arbitrary function $f(t)$ with a Laplace transform $F(s)$, then we obtain the following:

$$\mathcal{L}\left\{e^{at} f(t)\right\}(s) = \int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)} f(t)\, dt = \mathcal{L}\left\{f(t)\right\}(s-a) = F(s-a).$$

This is an important observation, which usually is called the **first translation formula** for the Laplace transform:

$$\boxed{\mathcal{L}\left\{e^{at} f(t)\right\}(s) = F(s-a).} \tag{9}$$

In words, this formula says that to compute the Laplace transform of $f(t)$ multiplied by e^{at}, then it is only necessary to take the Laplace transform of $f(t)$ (namely, $F(s)$) and replace the variable s by $s - a$, where a is the coefficient of t in the exponential multiplier. Here is an example of this formula in use.

Example 2.1.9. Compute the Laplace transform of $e^{at} \sin bt$ and $e^{at} \cos bt$.

▶ **Solution.** From Example 2.1.7 we know that

$$\mathcal{L}\{\cos bt\}(s) = \frac{s}{s^2+b^2} \quad \text{and} \quad \mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2+b^2}.$$

Replacing s by $s-a$ in each of these formulas gives

$$\boxed{\mathcal{L}\left\{e^{at}\cos bt\right\}(s) = \frac{s-a}{(s-a)^2+b^2} \quad \text{and} \quad \mathcal{L}\left\{e^{at}\sin bt\right\}(s) = \frac{b}{(s-a)^2+b^2}.} \tag{10}$$

◀

For a numerical example, note that

$$\mathcal{L}\left\{e^{-t}\sin 3t\right\} = \frac{3}{(s+1)^2+9} \quad \text{and} \quad \mathcal{L}\left\{e^{3t}\cos\sqrt{2}t\right\} = \frac{s-3}{(s-3)^2+2}.$$

Example 2.1.10. Compute the Laplace transform of the functions $t^n e^{at}\cos bt$ and $t^n e^{at}\sin bt$.

▶ **Solution.** If $\alpha = a+bi$ then Euler's formula shows that $e^{\alpha t} = e^{at}(\cos bt + i\sin bt)$ so that multiplying by t^n gives

$$t^n e^{\alpha t} = t^n e^{at}\cos bt + it^n e^{at}\sin bt.$$

That is, $t^n e^{at}\cos bt$ is the real part and $t^n e^{at}\sin bt$ is the imaginary part of $t^n e^{\alpha t}$. Since

$$\mathcal{L}\left\{t^n e^{at}\cos bt\right\}(s) + i\mathcal{L}\left\{t^n e^{at}\sin bt\right\}(s) = \mathcal{L}\left\{t^n e^{\alpha t}\right\}(s) = \frac{n!}{(s-\alpha)^{n+1}} = \frac{n!}{(s-(a+bi))^{n+1}},$$

we conclude that

$$\boxed{\begin{aligned} \mathcal{L}\{t^n e^{at}\cos bt\}(s) &= \operatorname{Re}\left(\frac{n!}{(s-(a+bi))^{n+1}}\right) \quad \text{and} \\ \mathcal{L}\{t^n e^{at}\sin bt\}(s) &= \operatorname{Im}\left(\frac{n!}{(s-(a+bi))^{n+1}}\right). \end{aligned}} \tag{11}$$

◀

If we take $n = 1$ in the above example, then

$$\begin{aligned}
\frac{1}{(s-\alpha)^2} &= \frac{1}{((s-a)-ib)^2} \\
&= \frac{1}{((s-a)-ib)^2} \cdot \frac{((s-a)+ib)^2}{((s-a)+ib)^2} \\
&= \frac{((s-a)^2-b^2)+i2(s-a)b}{((s-a)^2+b^2)^2}.
\end{aligned}$$

By taking real and imaginary parts of this last expression, we conclude that

$$\mathcal{L}\left\{te^{at}\cos bt\right\}(s) = \frac{(s-a)^2-b^2}{((s-a)^2+b^2)^2} \quad \text{and} \quad \mathcal{L}\left\{te^{at}\sin bt\right\}(s) = \frac{2(s-a)b}{((s-a)^2+b^2)^2}. \qquad (12)$$

The functions that we have dealt with in this section occur repeatedly in the context of differential equations. To be able to speak succinctly of these functions, we shall say that the class $\mathcal{E}$ of **elementary functions** consists of all of the functions that can be written as sums of scalar multiples of the functions $t^n e^{at}\cos bt$ and $t^n e^{at}\sin bt$ for some integer $n \geq 0$ and real numbers a and b. Thus, the linearity theorem (Theorem 2.1.1) combined with the formulas (11) allow one to compute the Laplace transform of *any* elementary function.

Example 2.1.11. The following are typical elementary functions:

1.	$3t^2 + te^{-0.5t} + \frac{1}{2}\cos t$	2.	$e^t(t - 2\sin t)$
3.	$1 + t + t^2 + \cdots + t^n$	4.	$(t + e^{2t})^2$
5.	$\sin^2 t$	6.	$(1 + 3\cos t)(t - 4e^{t/3})$

The first three functions are clearly in the class $\mathcal{E}$. We will leave it as an exercise to check that the last three are also in $\mathcal{E}$.

The following are some typical functions that you might easily encounter, but which are *not* in the class $\mathcal{E}$ that we have labelled elementary functions.

1. $1/t$ 2. $\ln t$ 3. e^{t^2} 4. $\tan t$ 5. $\sqrt{t}$

Example 2.1.12. Compute the Laplace transform of $3 - 5\cos 2t + 2e^{3t}$.

► **Solution.** Using the formulas derived above and linearity we obtain

$$\begin{aligned}
\mathcal{L}\left\{3 - 5\cos 2t + 2e^{3t}\right\}(s) &= 3\mathcal{L}\left\{1\right\}(s) - 5\mathcal{L}\left\{\cos 2t\right\}(s) + 2\mathcal{L}\left\{e^{3t}\right\}(s) \\
&= \frac{3}{s} - \frac{5s}{s^2+4} + \frac{2}{(s-3)}
\end{aligned}$$

◄

Exercises

1. Compute the Laplace transform of each function given below directly from the integral definition given in Equation (1).

   (a) $3t + 1$

   ► **Solution.**

$$\begin{aligned}
\mathcal{L}\{3t+1\}(s) &= \int_0^\infty (3t+1)e^{-st}\,dt \\
&= 3\int_0^\infty te^{-st}\,dt + \int_0^\infty e^{-st}\,dt \\
&= 3\left(\frac{t}{-s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt\right) + \frac{-1}{s}e^{-st}\Big|_0^\infty \\
&= 3\left(\left(\frac{1}{s}\right)\left(\frac{-1}{s}\right)e^{-st}\Big|_0^\infty\right) + \frac{1}{s} \\
&= \frac{3}{s^2} + \frac{1}{s}.
\end{aligned}$$

◄

   (b) $5t - 9e^t$ (c) $e^{2t} - 3e^{-t}$ (d) te^{-3t}

2. Use linearity and the formulas for the Laplace transform of elementary functions to verify your answers in Exercise 1.

Using the formulas for the Laplace transform of the elementary functions and the theorem on linearity, compute the Laplace transform of each of the elementary functions in Exercises 3 – 22.

3. $5e^{2t}$
4. $3e^{-7t} - 7t^3$
5. $t^2 - 5t + 4$
6. $t^3 + t^2 + t + 1$
7. $e^{-3t} + 7e^{-4t}$
8. $e^{-3t} + 7te^{-4t}$
9. $\cos 2t + \sin 2t$
10. $e^t(t - \cos 2t)$

11. $e^{-t/3}\cos\sqrt{6}t$
12. $(t+e^{2t})^2$
13. $(\sqrt{2}+(0.123)t)e^{-(1.1)t}$
14. $5\cos 2t - 3\sin 2t + 4$
15. $e^{5t}(8\cos 2t + 11\sin 2t)$
16. $t^2\sin 2t$
17. $e^{-at} - e^{-bt}$ for $a \neq b$.
18. $\cos^2 bt$ (*Hint:* $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$)
19. $\sin^2 bt$
20. $\sin bt\cos bt$ *Hint:* Use an appropriate trigonometric identity.
21. $\cosh bt$ (Recall that $\cosh bt = (e^{bt}+e^{-bt})/2$.)
22. $\sinh bt$ (Recall that $\sinh bt = (e^{bt}-e^{-bt})/2$.)
23. Verify that the function $f(t) = e^{t^2}$ does not have a Laplace transform. That is, show that the improper integral that defines $F(s)$ does not converge for *any* value of s.
24. Determine which of the following functions are in the class $\mathcal{E}$ of elementary functions.
    (a) t^2e^{-2t}
    (b) $t^{-2}e^{2t}$
    (c) t/e^t
    (d) e^t/t
    (e) $(t+e^t)^2$
    (f) $(t+e^t)^{-2}$
    (g) $te^{t/2}$
    (h) $t^{1/2}e^t$
    (i) $\sin 2t/e^{2t}$
    (j) $e^{2t}/\sin 2t$
25. Verify that the class of elementary functions $\mathcal{E}$ is **closed** under the following operations.
    (a) Addition. That is, show that if f and g are in $\mathcal{E}$, then so is $f+g$.
    (b) Multiplication. That is, show that if f and g are in $\mathcal{E}$, then so is fg.
    (c) Differentiation. That is, show that if f is in $\mathcal{E}$, then so is the derivative f'.

Show that $\mathcal{E}$ is *not closed* under the operation of multiplicative inverse. That is, find a function f in $\mathcal{E}$ such that $1/f$ is not in $\mathcal{E}$.

2.2 Inverse Laplace Transform

In the previous section we introduced the class of elementary functions $\mathcal{E}$. An elementary function $f(t)$ is one which can be obtained by taking sums of scalar multiples of functions of the form $t^n e^{at}\cos bt$ and $t^n e^{at}\sin bt$ for some choices of the integer $n \geq 0$ and real numbers a and b. Formulas (8) and (11), in conjunction with the linearity theorem (Theorem 2.1.1) provide the ability to compute the Laplace transform of *any* elementary function. A review of each of these formulas then shows that the Laplace transform of an elementary function is a sum of scalar multiples of terms

$$\mathrm{Re}\left(\frac{n!}{(s-\alpha)^{n+1}}\right) \quad \text{and} \quad \mathrm{Im}\left(\frac{n!}{(s-\alpha)^{n+1}}\right). \tag{1}$$

where $\alpha = a + bi$. We can write

$$\frac{n!}{(s-a-bi)^{n+1})} = \frac{n!}{(s-a-bi)^{n+1}} \cdot \frac{(s-a+bi)^{n+1}}{(s-a+bi)^{n+1}} = \frac{n!(s-a+bi)^{n+1}}{((s-a)^2+b^2)^{n+1}},$$

and then expand the numerator into powers of s. Since $i^{2k} = (-1)^k$ and $i^{2k+1} = (-1)^k i$ it follows that both parts of formula (1) are of the form

$$\frac{P(s)}{Q(s)} \tag{2}$$

where $P(s)$ is a (real) polynomial in the variable s of degree $\leq n+1$ and $Q(s) = ((s-a)^2+b^2)^{n+1}$ is a polynomial in s of degree $2n+2$. Recall that a **polynomial** is a function $f(s)$ of the form $f(s) = a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0$ where m is a nonnegative integer, and the coefficients a_j are real numbers. The **degree** of $f(s)$ is m if $a_m \neq 0$. A function $P(s)/Q(s)$ which is the quotient of two (real) polynomials is referred to as a (real) **rational function**. If the degree of the numerator is less than the degree of the denominator, then $P(s)/Q(s)$ is a **proper rational function**. The set of rational functions (with real coefficients) will be denoted by $\mathbb{R}(s)$ and the set of proper rational functions will be denoted by $\mathbb{R}_{\mathrm{pr}}(s)$.

Example 2.2.1. Among the following functions,

$$1.\ \frac{1}{s^2+4} \qquad 2.\ \frac{1}{s}+\frac{3}{s^2} \qquad 3.\ s^2+4 \qquad 4.\ \frac{s^2+3s-1}{3s^3+2s+5} \qquad 5.\ \frac{s^3+1}{s^2-1} \qquad 6.\ \frac{1}{s^{1/2}},$$

functions 1–5 are rational functions, that is in the set $\mathbb{R}(s)$, while functions 1, 2, and 4 are proper rational functions, that is, they are in the set $\mathbb{R}_{\text{pr}}(s)$. Function 6 is not rational since the exponent in the denominator is 1/2, which is not even an integer.

Example 2.2.2. Let $F_1(s) = 1/(s+1)$ and $F_2(s) = s/(s^2+4)$. Note that each of these rational functions is proper because the degree of the polynomial in the denominator is larger than the degree in the numerator. Computing $F_1(s) + F_2(s)$ and $F_1(s)F_2(s)$ gives

$$\begin{aligned} F_1(s) + F_2(s) &= \frac{1}{s+1} + \frac{s}{s^2+4} = \frac{2s^2+s+4}{s^3+s^2+4s+4} \\ F_1(s)F_2(s) &= \frac{1}{s+1}\frac{s}{s^2+4} = \frac{s}{s^3+s^2+4s+4}. \end{aligned}$$

What we observe is that $F_1(s)+F_2(s)$ and $F_1(s)F_2(s)$ are also proper rational functions. That is, the property of being a proper rational function is preserved under the algebraic operations of addition and multiplication. This is not a special property of the particular functions written down here, but is a general property of proper rational functions. This property is normally expressed by saying that the set $\mathbb{R}_{\text{pr}}(s)$ of proper rational functions is *closed under the operations of addition and multiplication.* We note that it is also closed under multiplication by scalars, that is, if $F_(s)$ is a proper rational function and a is a real constant, then $aF(s)$ is also a proper rational function.

The calculations of Laplace transforms of elementary functions done in the previous section, culminating in formulas (8) and (11), show that the Laplace transform is an operator which takes a function $f(t)$ in $\mathcal{E}$ and produces a function $F(s)$ in $\mathbb{R}_{\text{pr}}(s)$. In symbols, we have that $\mathcal{L}$ is a function

$$\mathcal{L} : \mathcal{E} \longrightarrow \mathbb{R}_{\text{pr}}(s).$$

According to the theory of partial fraction decompositions, every proper rational function is a sum of scalar multiples of the **simple rational functions**

$$\frac{1}{(s+a)^k}, \quad \frac{1}{(s^2+as+b)^k} \quad \text{and} \quad \frac{s}{(s^2+as+b)^k}$$

for appropriate choices of the constants a and b. Since

$$\mathcal{L}\left\{\frac{t^{k-1}e^{-at}}{(k-1)!}\right\} = \frac{1}{(s+a)^k}, \tag{3}$$

we see that the simple rational function $1/(s+a)^k$ is in the image of the Laplace transform operator $\mathcal{L}$. It is also true that each of the other simple rational functions is in the image of $\mathcal{L}$. This fact will be verified in Section 2.5 after some additional techniques are developed. What this says, in conjunction with the linearity principle (Theorem 2.1.1), is that the Laplace transform operator $\mathcal{L} : \mathcal{E} \longrightarrow \mathbb{R}_{\text{pr}}(s)$ is an onto function, i.e., the image of $\mathcal{L}$ is *all* of $\mathbb{R}_{\text{pr}}(s)$ (that is, *every* $F(s) \in \mathbb{R}_{\text{pr}}(s)$ can be written as $\mathcal{L}\{f(t)\}$ for some $f(t) \in \mathcal{E}$). It is also true, although we will not verify it directly, that $\mathcal{L} : \mathcal{E} \longrightarrow \mathbb{R}_{\text{pr}}(s)$ is a one-to-one function (that is, $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \implies f(t) = g(t)$). Therefore, $\mathcal{L}$ has an inverse function $\mathcal{L}^{-1} : \mathbb{R}_{\text{pr}}(s) \longrightarrow \mathcal{E}$, that we will refer to as the **inverse Laplace transform**, determined by the standard property of an inverse function:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \iff \mathcal{L}\{f(t)\} = F(s). \tag{4}$$

Thus Equation (3) is equivalent to the statement

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^k}\right\} = \frac{1}{(k-1)!}t^{k-1}e^{-at}, \tag{5}$$

while the formulas

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2} \quad \text{and} \quad \mathcal{L}\left\{\frac{\sin bt}{b}\right\} = \frac{1}{s^2+b^2}$$

are equivalent to the inverse Laplace transform formulas

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\} = \frac{\sin bt}{b}. \tag{6}$$

Moreover, the linearity property of $\mathcal{L}$, namely

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\},$$

produces a corresponding linearity property for the inverse Laplace transform $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathcal{L}^{-1}\{F(s)\} + c_2\mathcal{L}^{-1}\{G(s)\}.$$

Here, c_1 and c_2 are arbitrary constants.

We record these observations in the following fundamental result.

Theorem 2.2.3. *The Laplace transform $\mathcal{L} : \mathcal{E} \longrightarrow \mathbb{R}_{\text{pr}}(s)$ is a one-to-one onto operator with an inverse function $\mathcal{L}^{-1} : \mathbb{R}_{\text{pr}} \longrightarrow \mathcal{E}$. Moreover, $\mathcal{L}^{-1}$ satisfies the linearity property*

$$\mathcal{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathcal{L}^{-1}\{F(s)\} + c_2\mathcal{L}^{-1}\{G(s)\},$$

where c_1 and c_2 are arbitrary constants.

The set $\mathcal{E}$ of elementary functions will be referred to as the **input space** for the Laplace transform and the set $\mathbb{R}_{\text{pr}}(s)$ of proper rational functions will be referred to as the **transform space**. Thus, we think of the Laplace transform dynamically as transforming a function $f(t)$ from the input space $\mathcal{E}$ into a function $F(s)$ in the transform space $\mathbb{R}_{\text{pr}}(s)$. Similarly, we think of the inverse Laplace transform dynamically as transforming a function $F(s)$ in the transform space into a function $f(t)$ in the input space. A pair of functions $f(t) \in \mathcal{E}$ and $F(s) \in \mathbb{R}_{\text{pr}}(s)$ related by $\mathcal{L}\left\{f(t)\right\} = F(s)$ is called a **transform pair**, and we will express this relationship by means of the symbol $f(t) \longleftrightarrow F(s)$. Of course, we already have the convention that the Laplace transform of a function $f(t)$, named by a lower case letter, is indicated by the corresponding upper case letter $F(s)$. But the notation of transform pairs is particularly suited to functions indicated by explicit formulas. Thus,

$$1 \longleftrightarrow \frac{1}{s}, \qquad e^{3t} \longleftrightarrow \frac{1}{s-3}, \quad \text{and} \quad \sin t \longleftrightarrow \frac{1}{s^2+1}$$

are examples of transform pairs.

In applications it is commonly necessary to be able to find $f(t)$ given $F(s)$. For example, it is frequently easy to produce the Laplace transform $Y(s)$ of a solution of a differential equation. Then to solve the equation, it is necessary to find $y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\}$. This particular application will be explored in Section 2.4 and in more detail in later chapters.

One technique for finding $f(t)$ given $F(s)$ is to assemble a table of Laplace transform pairs, such as we have done in Table C.2, and try to manipulate $F(s)$ so that it is possible to recognize it in the table. Since $F(s)$ is a proper rational function, at least in the cases considered in this chapter, the manipulation of $F(s)$ will consist primarily of the algebraic tool of partial fraction decomposition, which you have probably studied previously in your calculus class. Partial fraction decompositions as needed for inverse Laplace transform calculations will be considered in detail in Section 2.3. For now, we will illustrate this technique with a few simple examples after first recalling the first translation formula and expressing it in the language of inverse Laplace transforms.

In the language of transform pairs, the first translation formula (Equation (9) of Section 2.1) can be expressed as follows:

Translation in transform space

$$e^{at}f(t) \quad \longleftrightarrow \quad F(s-a)$$

Since $\mathcal{L}^{-1}\{F(s)\} = f(t)$, this formula states that

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t) = e^{at}\mathcal{L}^{-1}\{F(s)\}.$$

Multiplying both sides of this equation by e^{-at}, we arrive at the following formula, which we will refer to as the **alternate first translation formula**:

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = e^{-at}\mathcal{L}^{-1}\{F(s-a)\}} \tag{7}$$

Here are some examples of the use of this formula.

Example 2.2.4. Compute $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{(s+1)^2}\right\}$.

► **Solution.** Let $F(s) = \dfrac{2s+3}{(s+1)^2}$. The strategy is to try to choose a constant a so that the translated function $F(s-a)$ becomes recognizable among the functions whose inverse Laplace transforms we have already identified in Equations (5) and (6). If we let $a = 1$ (chosen to simplify the denominator) then

$$F(s-a) = F(s-1) = \frac{2(s-1)+3}{((s-1)+1)^2} = \frac{2s+1}{s^2} = \frac{2}{s} + \frac{1}{s^2}.$$

Applying the alternate first translation formula and the fact that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^k}\right\} = \frac{t^{k-1}}{(k-1)!}$$

(Equation (5)) we conclude

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} = e^{-t}\mathcal{L}^{-1}\{F(s-1)\} \\ &= e^{-t}\left(\mathcal{L}^{-1}\left\{\frac{2}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\right) \\ &= e^{-t}(2+t). \end{aligned}$$

◄

Example 2.2.5. Compute $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+6s+25}\right\}$.

▶ **Solution.** Start by completing the square in the denominator to get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+25}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+16}\right\}.$$

The right hand side is in precisely the form of Equation (10) of Section 2.1, so we conclude that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+25}\right\} = \frac{e^{-3t}\sin 4t}{4}.$$

◀

Example 2.2.6. Compute $\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+4s+7}\right\}$.

▶ **Solution.** Let $F(s) = \frac{3s+2}{s^2+4s+7}$, and start by completing the square to get

$$\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+4s+7}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+2}{(s+2)^2+3}\right\}.$$

Letting $a = 2$ gives

$$F(s-a) = F(s-2) = \frac{3(s-2)+2}{((s-2)+2)^2+3} = \frac{3s-4}{s^2+3} = \frac{3s-4}{s^2+(\sqrt{3})^2}.$$

The alternate first translation formula and formulas (6) give

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+4s+7}\right\} &= e^{-2t}\mathcal{L}^{-1}\left\{\frac{3s-4}{s^2+(\sqrt{3})^2}\right\} \\
&= e^{-2t}\left(3\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{3})^2}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s^2+(\sqrt{3})^2}\right\}\right) \\
&= e^{-2t}\left(3\cos\sqrt{3}t - \frac{4}{\sqrt{3}}\sin\sqrt{3}t\right).
\end{aligned}$$

◀

Example 2.2.7. Compute $\mathcal{L}^{-1}\left\{\frac{5s+11}{s^2+6s-7}\right\}$.

▶ **Solution.** The difference between this example and the previous one is that the denominator factors as $s^2+6s-7 = (s+7)(s-1)$. Using the technique of partial fraction decomposition that you learned in calculus (and which will be reviewed in detail in the next section) we get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{5s+11}{s^2+6s-7}\right\} &= \mathcal{L}^{-1}\left\{\frac{3}{s+7}+\frac{2}{s-1}\right\} \\ &= 3\mathcal{L}^{-1}\left\{\frac{1}{s+7}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\ &= 3e^{-7t}+2e^{t}.\end{aligned}$$

◀

Exercises

1. Identify each of the following functions as proper rational **(PR)**, rational but not proper rational **(R)** , or not rational **(NR)**.

(a) $\dfrac{s^2-1}{(s-2)(s-3)}$ (b) $\dfrac{2s-1}{(s-2)(s-3)}$ (c) s^3-s^2+1

(d) $\dfrac{1}{s-2}+\dfrac{s}{(s+1)(s^2+1)}$ (e) $\dfrac{1}{s-2}\cdot\dfrac{s}{(s+1)(s^2+1)}$ (f) $\dfrac{2s+4}{s^{3/2}-s+1}$

(g) $\dfrac{\cos(s+1)}{\sin(s^2+1)}$ (h) $\left(\dfrac{3s-4}{2s^2+s+5}\right)^2$ (i) $\dfrac{2^s}{3^s}$

In Exercises 2 through 20 compute $\mathcal{L}^{-1}\{F(s)\}$ for the given proper rational function $F(s)$.

2. $\dfrac{-5}{s}$

3. $\dfrac{3}{s^2}-\dfrac{4}{s^3}$

4. $\dfrac{4}{2s+3}$

5. $\dfrac{3s}{s^2+2}$

6. $\dfrac{-2s}{3s^2+2}$

7. $\dfrac{2}{s^2+3}$

8. $\dfrac{3s+2}{3s^2+2}$

9. $\dfrac{1}{s^2+6s+9}$

10. $\dfrac{2s-5}{s^2+6s+9}$

11. $\dfrac{2s-5}{(s+3)^3}$

12. $\dfrac{2s^2-5s+1}{(s-2)^4}$

13. $\dfrac{s+2}{(s+2)^2+9}$

14. $\dfrac{s-1}{s^2-2s+10}$

15. $\dfrac{2s+5}{s^2+6s+18}$

16. $\dfrac{3s-2}{s^2+4s+6}$

17. $\dfrac{5s+3}{2s^2+2s+1}$

18. $\dfrac{s}{s^2-5s+6}$

19. $\dfrac{5}{s^2+2s-8}$

20. $\dfrac{2s+6}{s^2-6s+5}$

21. Verify each of the following inverse Laplace transform formulas:

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+a)^2+b^2}\right\} = \dfrac{e^{-at}\sin bt}{b}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+a)^2+b^2}\right\} = \dfrac{e^{-at}(b\cos bt - a\sin bt)}{b}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+a)^2-b^2}\right\} = \dfrac{e^{-at}\sinh bt}{b}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+a)^2-b^2}\right\} = \dfrac{e^{-at}(b\cosh bt - a\sinh bt)}{b}$

2.3 Partial Fractions

The Laplace transform of any elementary function $f(t) \in \mathcal{E}$ is a proper rational function $F(s) = P(s)/Q(s)$. The problem of finding $f(t)$ given $F(s)$, that is, finding $\mathcal{L}^{-1}\{F(s)\}$ is facilitated by writing $F(s)$ as a sum of simpler proper rational functions, known as the **partial fractions** of $F(s)$. The process of writing a given proper rational function $F(s)$ as a sum of partial fractions is known as the **partial fraction decomposition** of $F(s)$. The partial fractions are chosen from the **simple rational functions**

$$\frac{1}{(s-r)^k}, \quad \frac{1}{(s^2+bs+c)^k} \quad \text{and} \quad \frac{s}{(s^2+bs+c)^k},$$

where r, b and c are real numbers and the quadratic s^2+bs+c has no real roots, i.e., s^2+bs+c is irreducible over the reals. Since the roots of s^2+bs+c are $(-b\pm\sqrt{b^2-4c})/2$, the roots are not real precisely when $b^2-4c<0$.

Since $F(s)$ is a proper rational function, it may be written as

$$F(s) = \frac{P(s)}{Q(s)} = \frac{b_m s^m + b_{m-1}s^{n-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \tag{1}$$

where $n > m$. We will always assume that the coefficient of the highest term in $Q(s)$ is 1, so that

$$Q(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0.$$

(If this is not the case, then one has to factor out the leading coefficient a_n of the denominator, $Q(s)$ before starting with the partial fraction decomposition). The denominator $Q(s)$, which is a polynomial of degree n, will have a certain number of real roots, and a certain number of roots $\alpha \in \mathbb{C}$ (but $\alpha \notin \mathbb{R}$) that appear in complex conjugate pairs α and $\overline{\alpha}$. If $\alpha = \beta + i\gamma$ then

$$(s-\alpha)(s-\overline{\alpha}) = s^2 - 2\beta s + (\beta^2+\gamma^2).$$

Thus complex conjugate pairs of roots can be combined to give irreducible quadratic factors of $Q(s)$. Hence $Q(s)$ can be expressed as

$$Q(s) = (s-r_1)^{k_1}\cdots(s-r_h)^{k_h}(s^2+b_1s+c_1)^{l_1}\cdots(s^2+b_js+c_j)^{l_j}, \tag{2}$$

where $r_1, \ldots, r_h$ are h distinct real numbers, the j distinct real second order terms $s^2+b_1s+c_1, \ldots, s^2+b_js+c_j$ are irreducible, and $k_1+\cdots+k_h+2l_1+\cdots+2l_j = n$.

With these notational preliminaries out of the way, *the partial fraction decomposition of the proper rational function* $F(s) = P(s)/Q(s)$ *is a sum of exactly* $n = \deg(Q(s))$ *scalar multiples of simple rational functions determined from the denominator* $Q(s)$ *by the following two rules:*

PF1. If k is the power of a linear term $s - r$ in the factorization of $Q(s)$, then each of the following k terms appear in the partial fraction decomposition of $P(s)/Q(s)$:

$$\frac{1}{s-r}, \frac{1}{(s-r)^2}, \dots, \frac{1}{(s-r)^k}$$

PF2. If l is the power of an irreducible quadratic term $s^2 + bs + c$ in the factorization of $Q(s)$ then each of the following $2l$ terms appear in the partial fraction decomposition of $P(s)/Q(s)$:

$$\frac{1}{s^2+bs+c}, \frac{1}{(s^2+bs+c)^2}, \dots, \frac{1}{(s^2+bs+c)^l}$$

and

$$\frac{s}{s^2+bs+c}, \frac{s}{(s^2+bs+c)^2}, \dots, \frac{s}{(s^2+bs+c)^l}$$

By the **form** of the partial fraction decomposition for $P(s)/Q(s)$ we mean the expression of $P(s)/Q(s)$ as a linear combination, with undetermined coefficients, of the functions listed above corresponding to the roots (both real and complex) of the denominator $Q(s)$. Of course, one must find the coefficients, which generally involves solving some system of linear equations. The solution of the linear equations can be greatly simplified in certain commonly occurring special cases, which we explain in more detail now.

Case 1. Distinct real roots.

If $Q(s) = (s - r)Q_1(s)$, where r is a real number, $Q_1(r) \neq 0$ (so that r is a root of the denominator $Q(s)$ of multiplicity 1), and if $\deg P(s) < n$, then rule PF1 states that the term $1/(s - r)$ will appear in the partial fraction decomposition of $F(s) = P(s)/Q(s)$, but no higher power of $1/(s - r)$ will appear. Thus

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A}{s-r} + F_1(s), \tag{3}$$

where A is a constant to be determined, and $F_1(s)$ is a rational function representing all the terms not involving a power of $1/(s - r)$. In particular, $F_1(s)$ is defined at $s = r$ and $F_1(r) \neq 0$. Multiplying Equation (3) by $Q(s)$ gives

$$P(s) = A\frac{Q(s)}{s-r} + Q(s)F_1(s).$$

Since $Q(r) = 0$ we can rewrite this equation as

$$P(s) = A\frac{Q(s) - Q(r)}{s - r} + Q(s)F_1(s).$$

The first part of this expression is a difference quotient for computing the derivative of $Q(s)$, so taking the limit as s approaches r, gives $P(r) = AQ'(r)$. Thus, we have shown that the coefficient, in the partial fraction expansion, of a term $1/(s - r)$ corresponding to a simple root of the denominator $Q(s)$ is given by

$$\boxed{A = \frac{P(r)}{Q'(r)}.} \tag{4}$$

In the case that *all* of the roots of $Q(s)$ are distinct, so that

$$Q(s) = (s - r_1) \cdots (s - r_n),$$

then the complete partial fraction decomposition of $F(s) = P(s)/Q(s)$ has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n}, \tag{5}$$

and the coefficients A_i are given by Equation (4) as

$$\boxed{A_i = \frac{P(r_i)}{Q'(r_i)} \quad \text{where} \quad Q'(r_i) = \prod_{j \neq i}(r_i - r_j).} \tag{6}$$

The formula for $Q'(r_i)$ given in Equation (6), when written out without the summation sign is:

$$Q'(r_i) = (r_i - r_1) \cdots (r_i - r_{i-1})(r_i - r_{i+1}) \cdots (r_i - r_n).$$

In words, this says that $Q'(r_i)$ *is obtained from* $Q(s)$ *by deleting the term* $(s - r_i)$ *from* $Q(s)$ *and then replacing* s *with* r_i. For example, if $Q(s) = (s - 1)(s - 3)(s - 5)(s - 7)$, then

$$Q'(3) = (3 - 1)(3 - 5)(3 - 7) = 16.$$

Example 2.3.1. Find the partial fraction decomposition of

$$\frac{2s + 4}{s^2 - 2s - 3}.$$

▶ **Solution.** Here $Q(s) = s^2 - 2s - 3 = (s-3)(s+1)$ has roots 3 and -1 and

$$\frac{P(3)}{Q'(3)} = \frac{10}{4} = \frac{5}{2} \quad \text{and} \quad \frac{P(-1)}{Q'(-1)} = \frac{2}{-4} = -\frac{1}{2}.$$

Thus

$$\frac{2s+4}{s^2-2s-3} = \frac{5}{2} \cdot \frac{1}{s-3} - \frac{1}{2} \cdot \frac{1}{s+1}.$$

◀

Since $\mathcal{L}^{-1}\left\{\frac{1}{s-r}\right\} = e^{rt}$, the partial fraction expansion formula (5) immediately gives the following result, known as the Heaviside expansion formula.

Theorem 2.3.2 (Heaviside Expansion Formula). *If*

$$Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n),$$

where $r_1, \ldots, r_n$ *are distinct real numbers, and if* $\deg P(s) < n$*, then the inverse Laplace transform of* $F(s) = P(s)/Q(s)$ *is*

$$\boxed{\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(r_1)}{Q'(r_1)}e^{r_1 t} + \cdots + \frac{P(r_n)}{Q'(r_n)}e^{r_n t}.} \tag{7}$$

Example 2.3.3. Compute the inverse Laplace transform of

$$F(s) = \frac{s}{(s+1)(s-2)(s+3)}.$$

▶ **Solution.** We observe that the denominator is the product of distinct linear terms, so Theorem 2.3.2 applies and we obtain

$$\begin{aligned}\mathcal{L}^{-1}\{F(s\} &= \frac{(-1)e^{-t}}{(-1-2)(-1+3)} + \frac{(2)e^{2t}}{(2+1)(2+3)} + \frac{(-3)e^{-3t}}{(-3+1)(-3-2)} \\ &= \frac{e^{-t}}{6} + \frac{2e^{2t}}{15} - \frac{3e^{-3t}}{10}\end{aligned}$$

◀

Example 2.3.4. Find the inverse Laplace transform of

$$F(s) = \frac{s^2-4s+1}{s^3-9s}.$$

▶ **Solution.** Write $F(s) = P(s)/Q(s)$ and observe that $Q(s) = s(s-3)(s+3)$ factors into a product of distinct linear terms. Apply Theorem 2.3.2 to obtain

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \frac{P(0)e^{0t}}{(3)(-3)} + \frac{P(-3)e^{-3t}}{(-3)(-6)} + \frac{P(3)e^{3t}}{(3)(6)} \\ &= -\frac{1}{9} + \frac{22e^{-3t}}{18} - \frac{2e^{3t}}{18}\end{aligned}$$

◀

Case 2. Real roots of multiplicity > 1.

If $Q(s) = (s-r)^k Q_1(s)$ where $Q_1(r) \neq 0$ (that is r is a root of the denominator $Q(s)$ of multiplicity exactly k) and $\deg P(s) < n = \deg Q(s)$, then rule PF1 gives

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{s-r} + \frac{A_2}{(s-r)^2} + \cdots + \frac{A_k}{(s-r)^k} + F_1(s), \tag{8}$$

where $A_1, \ldots, A_k$ are constants to be determined and $F_1(s)$ is a rational function representing all the terms not involving a power of $1/(s-r)$. The constants can be determined by solving a system of linear equations as was done in calculus, or one can proceed as follows. Multiply Equation (8) by $(s-r)^k$ to clear the denominators of powers of $s-r$. This gives

$$(s-r)^k F(s) = A_1(s-r)^{k-1} + A_2(s-r)^{k-2} + \cdots + A_k + (s-r)^k F_1(s). \tag{9}$$

Since $F_1(r) \neq 0$, if we let $H(s) = (s-r)^k F_1(s)$ then it is a simple exercise using the product rule for derivatives to conclude that $H^{(j)}(r) = 0$ for $0 \leq j \leq k-1$. Letting $G(s) = (s-r)^k F(s)$ and applying this observation to Equation (9), we conclude

$$\boxed{A_j = \frac{G^{(k-j)}(r)}{(k-j)!}.} \tag{10}$$

To see this, note that

$$\left.\frac{d^\ell}{ds^\ell}(s-r)^m\right|_{s=r} = \begin{cases} 0 & \text{if } 0 \leq \ell < m, \\ m! & \text{if } \ell = m \\ 0 & \text{if } \ell > m, \end{cases}$$

and apply this observation to the $(k-j)^{\text{th}}$ derivative of (9) to arrive at (10).

Remark 2.3.5. For polynomials, the calculation of the Taylor polynomial centered at r is easily accomplished algebraically (without formally computing derivatives) by means of the substitution $s = (s-r)+r$. Thus if $P(s) = s^2+4s-3$ and $r = 2$, the Taylor polynomial centered about $r = 2$ is given by

$$P(s) = ((s-2)+2)^2 + 4((s-2)+2) - 3 = (s-2)^2 + 8(s-2) + 9.$$

Example 2.3.6. Find the partial fraction decomposition of $\dfrac{s^2+4s-3}{(s-2)^3}$.

▶ **Solution.** As observed above, the Taylor series for $P(s) = s^2+4s-3$ about $r = 2$ is given by

$$s^2+4s-3 = (s-2)^2 + 8(s-2) + 9.$$

Dividing by $(s-2)^3$ gives

$$\frac{s^2+4s-3}{(s-2)^3} = \frac{1}{s-2} + 8\cdot\frac{1}{(s-2)^2} + 9\cdot\frac{1}{(s-2)^3}.$$

◀

If $Q(s) = (s-r_1)^{k_1}\cdots(s-r_h)^{k_h}$ is a polynomial of degree $n = k_1+\cdots+k_h$ and $\deg P(s) < n$, then $P(s)/Q(s)$ will be a sum of h parts, and each of these h parts will have the form of Equation (8) with n replaced by $k_1, k_2, \ldots, k_h$. In this case one should work directly with systems of linear equations to find the constants A_j, as illustrated by the following example.

Example 2.3.7. Find the partial fraction decomposition of

$$F(s) = \frac{s^3+s+3}{(s-1)^3(s+2)}.$$

▶ **Solution.** The partial fraction expansion of $F(s)$ can be written as

$$\frac{P(s)}{Q(s)} = \frac{s^3+s+3}{(s-1)^2(s+2)} = \frac{A_1}{s-1} + \frac{A_2}{(s-1)^2} + \frac{A_3}{(s-1)^3} + \frac{B}{s+2}.$$

If we multiply both sides of this equation by $Q(s) = (s-1)^3(s+2)$ we get

$$P(s) = s^3+s+3 = A_1(s-1)^2(s+2) + A_2(s-1)(s+2) + A_3(s+2) + B(s-1)^3.$$

If we set $s = 1$, we conclude that $A_3 = P(1)/3 = 5/3$; if we set $s = -2$, we conclude that $B = P(-2)/(-27) = 1/27$; and if we compare the coefficients of s^3 on the left and right of this equation, we see that $A_1 + B = 1$, so $A_1 = 26/27$. Thus we have determined all of the coefficients except for A_2. This can be determined by evaluating both sides of the above equation at another value of s. Any value of s not already used will work, so we may as well choose a simple one such as $s = 0$. This gives $P(0) = 3 = 2A_1 - 2A_2 + 2A_3 - B$. Solving for A_2 gives $A_2 = 10/9$. Hence,

$$F(s) = \frac{26}{27} \cdot \frac{1}{s-1} + \frac{10}{9} \cdot \frac{1}{(s-1)^2} + \frac{5}{3} \cdot \frac{1}{(s-1)^3} + \frac{1}{27} \cdot \frac{1}{s+2}.$$

◀

Example 2.3.8. Compute $\mathcal{L}^{-1}\{F(s)\}$ for the proper rational function

$$F(s) = \frac{s^3 + s + 3}{(s-1)^3(s+2)}.$$

▶ **Solution.** This is the function $F(s)$ whose partial fraction expansion was computed in the previous example as

$$F(s) = \frac{26}{27} \cdot \frac{1}{s-1} + \frac{10}{9} \cdot \frac{1}{(s-1)^2} + \frac{5}{3} \cdot \frac{1}{(s-1)^3} + \frac{1}{27} \cdot \frac{1}{s+2}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-r)^k}\right\} = \frac{1}{(k-1)!} t^{k-1} e^{rt},$$

we conclude that

$$\mathcal{L}^{-1}\{F(s)\} = \frac{26}{27} e^t + \frac{10}{9} t e^t + \frac{5}{6} t^2 e^t + \frac{1}{27} e^{-2t}.$$

◀

Case 3. Irreducible quadratic denominators.

If $Q(s) = (s^2 + as + b)^l$, where $s^2 + as + b$ is an irreducible real second order term and $\deg P(s) < 2l = n$, then

$$\frac{P(s)}{Q(s)} = \frac{B_1 s + C_1}{s^2 + as + b} + \frac{B_2 s + C_2}{(s^2 + as + b)^2} + \cdots + \frac{B_l s + C_l}{(s^2 + as + b)^l}, \tag{11}$$

where B_i, C_i are constants which can be determined by solving a system of linear equations as was done in calculus.

If

$$Q(s) = (s - r_1)^{k_1} \cdots (s - r_h)^{k_h} (s^2 + b_1 s + c_1)^{l_1} \cdots (s^2 + b_j s + c_j)^{l_j},$$

where $r_1, \ldots, r_h$ are h distinct real numbers, the j distinct real second order terms $s^2 + b_1 s + c_1, \ldots, s^2 + b_j s + c_j$ are irreducible, and $k_1 + \cdots + k_h + 2l_1 + \cdots + 2l_j = n$, then $P(s)/Q(s)$ will be a sum of $i + j$ parts, and each of the first i parts will have the form of Equation (8) with n replaced by $k_1, \cdots, k_h$, and each of the last j parts will have the form of Equation (11) with l replaced by $l_1, \cdots, l_j$. In this case one should work with systems of linear equations to find the constants A_r, B_r, C_r, as illustrated by the following example.

Example 2.3.9. (a) Find the partial fraction decomposition of

$$F(s) = \frac{s+3}{(s-1)^2(s^2+1)}.$$

▶ **Solution.** According to the above remarks,

$$\frac{s+3}{(s-1)^2(s^2+1)} = \frac{A_1}{s-1} + \frac{A_2}{(s-1)^2} + \frac{B_1 s + C_1}{s^2+1}.$$

If we multiply both sides of this equation by $Q(s) = (s-1)^2(s^2+1)$ we get

$$s + 3 = A_1(s-1)(s^2+1) + A_2(s^2+1) + (B_1 s + C_1)(s-1)^2.$$

If we set $s = 1$, we conclude that $A_2 = 2$; i.e.,

$$-2s^2 + s + 1 = A_1(s-1)(s^2+1) + (B_1 s + C_1)(s-1)^2.$$

To compute the three unknowns A_1, B_1, C_1 we select three numbers different from 1, for example $s = 0, -1, 2$, and obtain the three equations

$$\begin{array}{rcrcrcr} 1 & = & -A_1 & & & + & C_1 \\ -2 & = & -4A_1 & - & 4B_1 & + & 4C_1 \\ -5 & = & 5A_1 & + & 2B_1 & + & C_1, \end{array}$$

whose solutions are $A_1 = -3/2$, $B_1 = 3/2$, and $C_1 = -1/2$. Hence,

$$\frac{s+3}{(s-1)^2(s^2+1)} = -\frac{3}{2} \cdot \frac{1}{s-1} + 2 \cdot \frac{1}{(s-1)^2} + \frac{3}{2} \cdot \frac{s}{s^2+1} - \frac{1}{2} \cdot \frac{1}{s^2+1}.$$

◀

(b) Find the partial fraction decomposition of

$$\frac{P(s)}{Q(s)} = \frac{4s^2 - 16s}{(s^2+4)(s-2)^2}.$$

▶ **Solution.** The form of the decomposition is

$$\frac{4s^2-16s}{(s^2+4)(s-2)^2} = \frac{Bs+C}{s^2+4} + \frac{A_1}{s-2} + \frac{A_2}{(s-2)^2}.$$

Multiply both sides by the denominator $Q(s) = (s^2+4)(s-2)^2$ to obtain

$$4s^2-16s = (Bs+C)(s-2)^2 + A_1(s^2+4)(s-2) + A_2(s^2+4).$$

Multiply the right side out and gather coefficients to get

$$\begin{aligned} 4s^2-16s = \quad & (B+A_1)s^3 + (-4B+C-2A_1+A_2)s^2 \\ & +(4B-4C+4A_1)s + (4C-8A_1+4A_2). \end{aligned}$$

Equate the coefficients to obtain the following system of equations:

$$\begin{array}{rcrcrcrcr} B & & & + & A_1 & & & = & 0 \\ -4B & + & C & - & 2A_1 & + & A_2 & = & 4 \\ 4B & - & 4C & + & 4A_1 & & & = & -16 \\ & & 4C & - & 8A_1 & + & 4A_2 & = & 0 \end{array}$$

This system of linear equations can be solved by the standard Gauss-Jordan elimination technique. See Chapter 5 for details. The solution obtained is

$$B = -1, \quad C = 4, \quad A_1 = 1, \quad \text{and} \quad A_2 = -2,$$

which produces the partial fraction decomposition

$$\frac{4s^2-16s}{(s^2+4)(s-2)^2} = -\frac{s}{(s^2+4)} + 4\cdot\frac{1}{(s^2+4)} + \frac{1}{s-2} - 2\cdot\frac{1}{(s-2)^2}.$$

◀

(c) What is the **form** of the partial fraction decomposition of

$$\frac{P(s)}{Q(s)} = \frac{3s^2+2s-1}{(s^2+2s+2)^2(s-1)(s+4)^3}.$$

▶ **Solution.** The form of the partial fraction decomposition is

$$\frac{P(s)}{Q(s)} = \frac{B_1s+C_1}{(s^2+2s+2)} + \frac{B_2s+C_2}{(s^2+2s+2)^2} + \frac{A}{s-1} + \frac{D_1}{s+4} + \frac{D_2}{(s+4)^2} + \frac{D_3}{(s+4)^3}.$$

We observe that the form of the partial fraction decomposition is completely independent of the numerator $P(s)$. By multiplying both sides by the common denominator $Q(s)$, and equating the coefficients one is led to a system of eight equations that determine the eight coefficients $A, B_1, B_2, C_1, C_2, D_1, D_2, D_3$. ◀

Example 2.3.10. Compute the inverse Laplace transforms of the proper rational functions

$$F(s) = \frac{s+3}{(s-1)^2(s^2+1)} \quad \text{and} \quad G(s) = \frac{4s^2-16s}{(s^2+4)(s-2)^2}$$

from parts (a) and (b) of the previous example.

▶ **Solution.** Since

$$F(s) = -\frac{3}{2}\cdot\frac{1}{s-1} + 2\cdot\frac{1}{(s-1)^2} + \frac{3}{2}\cdot\frac{s}{s^2+1} - \frac{1}{2}\cdot\frac{1}{s^2+1},$$

we conclude that

$$\mathcal{L}^{-1}\left\{F(s)\right\} = -\frac{3}{2}e^t + 2te^t + \frac{3}{2}\cos t - \frac{1}{2}\sin t,$$

and

$$G(s) = -\frac{s}{(s^2+4)} + 4\cdot\frac{1}{(s^2+4)} + \frac{1}{s-2} - 2\cdot\frac{1}{(s-2)^2}$$

implies that

$$\mathcal{L}^{-1}\left\{G(s)\right\} = -\cos 2t + 2\sin 2t + e^{2t} - 2te^{2t}.$$

◀

Summary

By the algebraic technique of partial fraction decomposition, it is possible to write every proper rational function $F(s)$ as a linear combination of simple rational functions

$$\frac{1}{(s-r)^k}, \quad \frac{1}{(s^2+bs+c)^k} \quad \text{and} \quad \frac{s}{(s^2+bs+c)^k},$$

where r, b and c are real numbers and the quadratic s^2+bs+c is irreducible over the reals. Thus we can find the inverse Laplace transform of *any* proper rational function provided that we can find the inverse Laplace transform of the simple rational functions. But we have already seen that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-r)^k}\right\} = \frac{1}{(k-1)!}t^{k-1}e^{rt},$$

while in Section 2.2 we saw how to combine completion of the square with the first translation formula (Equation (7)) in order to compute the inverse Laplace transform of the simple rational functions

$$\frac{1}{(s^2+bs+c)^k} \quad \text{and} \quad \frac{s}{(s^2+bs+c)^k},$$

for the case $k = 1$. The general case for $k > 1$ will be considered in Section 2.5. For now we will show how to use the techniques already developed to handle the case $k = 2$. That is we will compute

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+bs+c)^2}\right\} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{s}{(s^2+bs+c)^2}\right\}.$$

By completing the square and using the first translation formula, it is sufficient to establish the following result.

Proposition 2.3.11. *We have the following formulas:*

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^2}\right\}(t) = \frac{1}{2b}t\sin bt \tag{12}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^2}\right\}(t) = \frac{1}{2b^3}(\sin bt - bt\cos bt). \tag{13}$$

Proof. Equations (12) of Section 2.1, namely

$$\mathcal{L}\left\{te^{at}\cos bt\right\}(s) = \frac{(s-a)^2-b^2}{((s-a)^2+b^2)^2} \quad \text{and} \quad \mathcal{L}\left\{te^{at}\sin bt\right\}(s) = \frac{2(s-a)b}{((s-a)^2+b^2)^2},$$

imply (by setting $a = 0$) that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^2}\right\} = \frac{1}{2b}t\sin bt \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{s^2-b^2}{(s^2+b^2)^2}\right\} = t\cos bt.$$

The first formula is the first of the required two formulas. For the second formula, note that

$$\frac{s^2-b^2}{(s^2+b^2)^2} = \frac{s^2+b^2}{(s^2+b^2)^2} - \frac{2b^2}{(s^2+b^2)^2} = \frac{1}{s^2+b^2} - \frac{2b^2}{(s^2+b^2)^2}.$$

Hence,

$$\frac{1}{(s^2+b^2)^2} = \frac{1}{2b^2}\left(\frac{1}{s^2+b^2} - \frac{s^2-b^2}{(s^2+b^2)^2}\right),$$

so that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^2}\right\} = \frac{1}{2b^2}\left(\frac{\sin bt}{b} - t\cos bt\right) = \frac{1}{2b^3}(\sin bt - bt\cos bt),$$

as required. ∎

Exercises

Use partial fraction decompositions to find the inverse Laplace transform of the given proper rational function.

1. $\dfrac{1}{(s+2)(s-5)}$

2. $\dfrac{5s+9}{(s-1)(s+3)}$

3. $\dfrac{8+s}{s^2-2s-15}$

4. $\dfrac{1}{s^2-3s+2}$

5. $\dfrac{5s-2}{s^2+2s-35}$

6. $\dfrac{3s+1}{s^2+s}$

7. $\dfrac{2s+11}{s^2-6s-7}$

8. $\dfrac{2s^2+7}{(s-1)(s-2)(s-3)}$

9. $\dfrac{s+1}{s^2-3}$

10. $\dfrac{s^2+s+1}{(s-1)(s^2+3s-10)}$

11. $\dfrac{7}{(s+4)^4}$

12. $\dfrac{s}{(s-3)^3}$

13. $\dfrac{s^2+s-3}{(s+3)^3}$

14. $\dfrac{5s^2-3s+10}{(s+1)(s+2)^2}$

15. $\dfrac{s^2-6s+7}{(s^2-4s-5)^2}$

16. $\dfrac{2}{(s+1)^2+16}$

17. $\dfrac{2s}{(s+1)^2+16}$

18. $\dfrac{5}{2s+3}$

19. $\dfrac{s+3}{4s^2+4s-3}$

20. $\dfrac{3s+2}{(s-2)^2+3}$

21. $\dfrac{2+3s}{s^2+6s+13}$

22. $\dfrac{5+2s}{s^2+4s+29}$

23. $\dfrac{3s+1}{(s-1)(s^2+1)}$

24. $\dfrac{3s^2-s+6}{(s+1)(s^2+4)}$

25. $\dfrac{2s^2+14}{(s-1)(s^2+2s+5)}$

26. $\dfrac{s^3+3s^2-s+3}{(s^2+4)^2}$

2.4 Initial Value Problems

The Laplace transform is particularly well suited for solving certain types of differential equations, namely the **constant coefficient linear differential equations**

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t), \tag{1}$$

where $a_0, \ldots, a_n$ are (real) constants, the function $f(t) \in \mathcal{E}$ is an elementary function, and the initial values of the unknown function $y(t)$ are also specified:

$$y(0) = y_0,\ y'(0) = y_1,\ \ldots,\ y^{(n-1)} = y_{n-1}.$$

Equation (1) with the initial values of the unknown function $y(t)$ specified is known as an **initial value problem**. The basic theory of this type of differential equation will be discussed in Chapter 3. For now, we will only study how the Laplace transform leads very quickly to a formula for $y(t)$.

The Laplace transform method for solving Equation (1) is based on the linearity property of the Laplace transform (Theorem 2.1.1) and the following formula which expresses the Laplace transform of the derivative an elementary function $f(t)$ as a simple algebraic function of $F(s) = \mathcal{L}\{f(t)\}$. Note that if $f(t) \in \mathcal{E}$ is an elementary function, then so is $f'(t)$. You were asked to verify this fact in Exercise 25 of Section 2.1.

Theorem 2.4.1. *Suppose $f(t) \in \mathcal{E}$ is an elementary function. Then $f'(t) \in \mathcal{E}$ and*

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0). \tag{2}$$

The following box summarizes the theorem in terms of transform pairs:

First derivative of input functions

$$f'(t) \longleftrightarrow sF(s) - f(0)$$

Proof. It has already been observed above that $f'(t) \in \mathcal{E}$. The formula (2) is obtained by applying integration by parts to the improper integral defining $\mathcal{L}\{f'(t)\}$, taking into account the convention that $g(t)|_0^\infty$ is a shorthand for $\lim_{t\to\infty}(g(t) - f(0))$, provided the limit exists. Applying integration by parts with $u = e^{-st}$ and $dv = f'(t)\,dt$ gives

$$\begin{aligned}
\mathcal{L}\{f'(t)\}(s) &= \int_0^\infty e^{-st} f'(t)\,dt \\
&= \left(f(t)e^{-st}\right)\Big|_0^\infty - \int_0^\infty (-s)e^{-st} f(t)\,dt \\
&= -f(0) + \int_0^\infty e^{-st} f(t)\,dt \\
&= s\mathcal{L}\{f(t)\}(s) - f(0).
\end{aligned}$$

The transition from the second to the third line is a result of the fact that functions $f(t) \in \mathcal{E}$ satisfy $\lim_{t\to\infty} f(t)e^{-st} = 0$, for s large. (See the discussion of limits on Page 78.) □

To avoid the notation becoming too heavy-handed, we will frequently write $\mathcal{L}\{f(t)\}$ rather than $\mathcal{L}\{f(t)\}(s)$. That is, the variable s may be suppressed when the meaning is clear. With this convention, Equation (2) becomes

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Example 2.4.2. Here are some simple examples of the validity of Equation (2).

1. If $f(t) = 1$, then $f'(t) = 0$ so $\mathcal{L}\{f'(t)\} = 0$, and
$$sF(s) - f(0) = s\frac{1}{s} - 1 = 0 = \mathcal{L}\{f'(t)\}.$$

2. If $f(t) = e^{at}$, then $f'(t) = ae^{at}$ so $\mathcal{L}\{f'(t)\} = \dfrac{a}{s-a}$ and
$$sF(s) - f(0) = \frac{s}{s-a} - 1 = \frac{a}{s-a} = \mathcal{L}\{f'(t)\}.$$

3. If $f(t) = \cos 3t$ then $f'(t) = -3\sin 3t$ so $\mathcal{L}\{f'(t)\} = -\dfrac{9}{s^2+9}$ and
$$sF(s) - f(0) = s\frac{s}{s^2+9} - 1 = -\frac{9}{s^2+9} = \mathcal{L}\{f'(t)\}.$$

Example 2.4.3. Solve the first order linear differential equation:

$$y' - 3y = 1, \quad y(0) = 1$$

▶ **Solution.** As is our convention, let $Y(s) = \mathcal{L}\{y(t)\}$. First compute the Laplace transform of each side of the equation. Using linearity of the Laplace transform (Theorem 2.1.1) and the differentiation formula (2) just verified, the left-hand side of the differential equation gives

$$\begin{aligned} \mathcal{L}\{y' - 3y\} &= \mathcal{L}\{y'\} - 3\mathcal{L}\{y\} \\ &= s\mathcal{L}\{y\} - 1 - 3\mathcal{L}\{y\} \\ &= (s-3)Y(s) - 1. \end{aligned}$$

For the right-hand side we have

$$\mathcal{L}\{1\} = \frac{1}{s}.$$

Equate these two expressions and solve for $Y(s)$ to get

$$Y(s) = \frac{1}{s-3}\left(1 + \frac{1}{s}\right) = \frac{1}{s-3} + \frac{1}{s(s-3)}.$$

A partial fraction decomposition applied to $\dfrac{1}{s(s-3)}$ gives

$$Y(s) = \frac{1}{s-3} + \frac{1}{3}\frac{1}{s-3} - \frac{1}{3}\frac{1}{s} = \frac{4}{3}\frac{1}{s-3} - \frac{1}{3}\frac{1}{s}.$$

Since $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ we can recover $y(t)$ from $Y(s)$ by the techniques of Section 2.3 to obtain

$$y(t) = \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = \frac{4}{3}e^{3t} - \frac{1}{3}.$$

◀

Let's consider another example.

Example 2.4.4. Solve the first order linear differential equation

$$y' + y = \sin t, \quad y(0) = 0.$$

▶ **Solution.** Letting $Y(s) = \mathcal{L}\{y(t)\}$, we equate the Laplace transform of each side of the equation to obtain

$$(s+1)Y(s) = \frac{1}{s^2+1}.$$

Solving for $Y(s)$ and decomposing $Y(s)$ into partial fractions gives

$$Y(s) = \frac{1}{2}\left(\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right).$$

Inversion of the Laplace transform gives

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right).$$

◀

If $f(t)$ is an elementary function, i.e., $f(t) \in \mathcal{E}$, then $f'(t)$ is also an elementary function so we may apply Theorem 2.4.1 with $f(t)$ replaced by $f'(t)$ (so that $(f')' = f''$) to get

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \\
&= s\left(s\mathcal{L}\{f(t)\} - f(0)\right) - f'(0) \\
&= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).
\end{aligned}$$

Thus we have arrived at the following formula for expressing the Laplace transform of $f''(t)$ in terms of $\mathcal{L}\{f(t)\}$ and the initial values $f(0)$ and $f'(0)$.

Corollary 2.4.5. *Suppose $f(t) \in \mathcal{E}$. Then*

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0). \tag{3}$$

The following box summarizes this result in terms of transform pairs:

Second derivative of input functions

$$f''(t) \quad \longleftrightarrow \quad s^2F(s) - sf(0) - f'(0)$$

The process used to determine the formula (3) for the Laplace transform of a second derivative can be repeated to arrive at a formula for the Laplace transform of the n^{th} derivative of an elementary function $f(t) \in \mathcal{E}$.

Theorem 2.4.6. *Suppose that $f(t) \in \mathcal{E}$ is an elementary function, and let $\mathcal{L}\{f(t)\} = F(s)$. Then*

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{4}$$

For $n = 3$ and $n = 4$, this formula becomes

$$\begin{aligned} \mathcal{L}\{f'''(t)\} &= s^3F(s) - s^2f(0) - sf'(0) - f''(0), \qquad \text{and} \\ \mathcal{L}\left\{f^{(4)}(t)\right\} &= s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0). \end{aligned}$$

If $f(0) = f'(0) = \cdots = f^{(n-1)} = 0$ then Equation (4) has the particularly simple form

$$\mathcal{L}\{f'(t)\} = s^nF(s).$$

In words, *the operation of differentiating n-times on the space of elementary functions with derivatives (up to order $n-1$) vanishing at 0, corresponds, under the Laplace transform, to the algebraic operation of multiplying by s^n on the space $\mathbb{R}_{\text{pr}}(s)$ of proper rational functions.*

We will now give several examples of how Equation (4) is used to solve some types of differential equations.

Example 2.4.7. Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

► **Solution.** As usual, let $Y(s) = \mathcal{L}\{y(t)\}$ and apply the Laplace transform to both sides of the differential equation to obtain

$$s^2Y(s) - 1 - Y(s) = 0.$$

Now solve for $Y(s)$ and decompose in partial fractions to get

$$Y(s) = \frac{1}{s^2-1} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}.$$

Then applying the inverse Laplace transform to $Y(s)$ gives

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}(e^t - e^{-t}).$$

◄

Example 2.4.8. Solve the initial value problem

$$y'' + 4y' + 4y = 2te^{-2t}, \quad y(0) = 1, \quad y'(0) = -3. \tag{5}$$

► **Solution.** Let $Y(s) = \mathcal{L}\{y(t)\}$ where, as usual, $y(t)$ is the unknown solution of Equation (5). Applying $\mathcal{L}$ to (5) gives the algebraic equation

$$s^2Y(s) - s + 3 + 4(sY(s) - 1) + 4Y(s) = \frac{2}{(s+2)^2},$$

which can be solved for $Y(s)$ to give

$$Y(s) = \frac{s+1}{(s+2)^2} + \frac{2}{(s+2)^4}. \tag{6}$$

Using the techniques of Section 2.3, (see Remark 2.3.5 in particular), write $s = (s+2)-2$ in the numerator of the first part to get

$$Y(s) = \frac{1}{s+2} - \frac{1}{(s+2)^2} + \frac{2}{(s+2)^4}.$$

Taking the inverse Laplace transform $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ then gives

$$y(t) = e^{-2t} - te^{-2t} + \frac{1}{3}t^2e^{-2t}.$$

◄

It is worth pointing out in this last example that in solving for $Y(s)$ we kept the part of $Y(s)$ that came from the initial values, namely $\frac{s+1}{(s+2)^2}$, distinct from that determined by the right-hand side of the equation, namely $\frac{2}{(s+2)^4}$. By not combining these into a single proper rational function before computing the partial fraction decomposition, we have simplified the computation of the partial fractions. This is a typical situation, and one that you should be aware of when working on exercises.

Example 2.4.9. Solve the initial value problem

$$y'' + \beta^2 y = \cos \omega t, \quad y(0) = y'(0) = 0,$$

where we assume that $\beta \neq 0$ and $\omega \neq 0$.

► **Solution.** Letting $Y(s) = \mathcal{L}\{y(t)\}$, applying $\mathcal{L}$ to the equation, and solving algebraically for $Y(s)$ gives

$$Y(s) = \frac{s}{(s^2+\beta^2)(s^2+\omega^2)}. \tag{7}$$

We will break our analysis into two cases: (1) $\beta^2 \neq \omega^2$ and (2) $\beta^2 = \omega^2$.

Case 1: $\beta^2 \neq \omega^2$.

In this case we leave it as an exercise to verify that the partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{1}{\omega^2-\beta^2}\left(\frac{s}{s^2+\beta^2} - \frac{s}{s^2+\omega^2}\right),$$

so that the solution $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ is

$$y(t) = \frac{\cos \beta t - \cos \omega t}{\omega^2 - \beta^2}.$$

Case 2: $\beta^2 = \omega^2$.

In this case

$$Y(s) = \frac{s}{(s^2+\omega^2)^2},$$

and formula (12) in Section 2.3 gives

$$y(t) = \frac{1}{2\omega} t \sin \omega t.$$

◄

Example 2.4.10. Solve the initial value problem

$$y''' - y'' + y' - y = 10e^{2t}, \quad y(0) = y'(0) = y''(0) = 0. \tag{8}$$

► **Solution.** Let $Y(s) = \mathcal{L}\{y(t)\}$ where $y(t)$ is the unknown solution to (8). Applying the Laplace transform $\mathcal{L}$ to (8) gives

$$s^3Y(s) - s^2Y(s) + sY(s) - Y(s) = \frac{10}{s-2}$$

which can be solved for $Y(s)$ to give

$$Y(s) = \frac{10}{(s^3 - s^2 + s - 1)(s-2)} = \frac{10}{(s-1)(s^2+1)(s-2)}.$$

Use the techniques of Section 2.3 to write $Y(s)$ in terms of its partial fractions:

$$Y(s) = \frac{-5}{s-1} + \frac{2}{s-2} + \frac{1+3s}{s^2+1}.$$

Taking the inverse Laplace transform $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ gives

$$y(t) = -5e^t + 2e^{2t} + \sin t + 3\cos t.$$

◄

We conclude this section by looking at what the Laplace transform tells us about the solution of the second order linear constant coefficient differential equation

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0,\ y'(0) = y_1, \tag{9}$$

where $f(t) \in \mathcal{E}$ is an elementary function, and a, b, and c are real constants. Applying the Laplace transform to Equation (9) (where $Y(s)$ is the Laplace transform of the unknown function $y(t)$, as usual) gives

$$a(s^2Y(s) - sy_0 - y_1) + b(sY(s) - y_0) + cY(s) = F(s).$$

If we let $P(s) = as^2 + bs + c$ ($P(s)$ is known as the **characteristic polynomial** of the differential equation), then the above equation can be solved for $Y(s)$ in the form

$$Y(s) = \frac{(as+b)y_0 + ay_1}{P(s)} + \frac{F(s)}{P(s)} = Y_1(s) + Y_2(s). \tag{10}$$

Notice that $Y_1(s)$ depends only on $P(s)$, which is determined by the left-hand side of the differential equation, and the initial values y_0 and y_1, while $Y_2(s)$ depends only on

$P(s)$ and the function $f(t)$ on the right-hand side of the equation. The function $f(t)$ is usually called the **input function** for the differential equation. Taking inverse Laplace transforms we can write

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{Y_1(s)\} + \mathcal{L}^{-1}\{Y_2(s)\} = y_1(t) + y_2(t).$$

The function $y_1(t)$ is the solution of (9) obtained by taking $f(t) = 0$, while $y_2(t)$ is the solution obtained by specifying that the initial conditions be zero, i.e., $y_0 = y_1 = 0$. Thus, $y_1(t)$ is referred to as the **zero-input solution**, while $y_2(t)$ is referred to as the **zero-state solution**. The terminology comes from engineering applications. A review of the examples above will show that the zero-state solution was computed in Examples 2.4.4, 2.4.9, and 2.4.10. You will be asked to compute further examples in the exercises, and addtional consequences of Equation (10) will be developed in Chapter 3.

Exercises

Use the Laplace transform to solve each of the following differential equations.

1. $y' + 6y = e^{3x}, \quad y(0) = 1$
2. $y' - 4y = 0, \quad y(0) = 2$
3. $y' - 4y = 3, \quad y(0) = 2$
4. $y' - 4y = t, \quad y(0) = 2$
5. $y' + 9y = 81t^2, \quad y(0) = -2$
6. $y' - 3y = \cos t, \quad y(0) = 0$
7. $y' + 2y = te^{-2t}, \quad y(0) = 0$
8. $y' - 3y = 50 \sin t, \quad y(0) = 1$
9. $y'' + 4y = 8, \quad y(0) = 2,\ y'(0) = 1$
10. $y'' - 3y' + 2y = 4, \quad y(0) = 2,\ y'(0) = 3$
11. $y'' - 3y' + 2y = e^t, \quad y(0) = -3,\ y'(0) = 0$
12. $y'' + 2y' - 3y = \sin 2t, \quad y(0) = 0,\ y'(0) = 0$
13. $y'' + 6y' + 9y = 50 \sin t, \quad y(0) = 0,\ y'(0) = 2$
14. $y'' + 25y = 0, \quad y(0) = 1,\ y'(0) = -1$

15. $y'' + 8y' + 16y = 0, \quad y(0) = \frac{1}{2},\ y'(0) = 2$
16. $y'' - 4y' + 4y = 4e^{2t}, \quad y(0) = -1,\ y'(0) = -4$
17. $y'' + y' + y = 0, \quad y(0) = 0,\ y'(0) = 1$
18. $y''' - y'' = t, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$
19. $y''' - y'' + y' - y = t, \quad y(0) = 0,\ y'(0) = 0,\ y''(0) = 0$
20. $y^{(4)} - y = 0, \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 0$
21. $y''' - y' = 6 - 3t^2, \quad y(0) = 1,\ y'(0) = 1,\ y''(0) = 1$

For each of the following differential equations, find the zero-state solution. Recall that the zero-state solution is the solution with all initial conditions equal to zero.

22. $y'' + 4y'13y = 0$
23. $y'' + 4y' + 3y = 6$
24. $y'' - y = \cos 3t$
25. $y'' + y = 4t \sin t$

2.5 Convolution

The Laplace transform $\mathcal{L} : \mathcal{E} \to \mathbb{R}_{\text{pr}}(s)$ provides a one-to-one linear correspondence between the input space $\mathcal{E}$ of elementary functions and the transform space $\mathbb{R}_{\text{pr}}(s)$ of proper rational functions. In the previous section we saw how an important operation on the functions in the input space $\mathcal{E}$, namely differentiation, corresponds to a natural algebraic operation on the transform space $\mathbb{R}_{\text{pr}}(s)$. Specifically, the formula is Theorem 2.4.1 which states that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0). \tag{1}$$

Our goal in this section is to study another operational identity of this type. Specifically, we will be concentrating on the question of what is the effect on the input space $\mathcal{E}$ of ordinary multiplication of functions in the transform space $\mathbb{R}_{\text{pr}}(s)$. Thus we are interested in the following question: Given functions $F(s)$ and $G(s)$ in $\mathbb{R}_{\text{pr}}(s)$ and their inverse Laplace transforms $f(t)$ and $g(t)$ in $\mathcal{E}$, what is the elementary function $h(t)$ such that $h(t)$ corresponds to $H(s) = F(s)G(s)$ under the Laplace transform? More precisely, how is $h(t)$ related to $f(t)$ and $g(t)$? In other words, how do we fill in the following question mark?

$$\boxed{\quad ? \quad} \longleftrightarrow F(s)G(s)$$

You might guess that $h(t) = f(t)g(t)$. That is, you would be guessing that multiplication in the input space corresponds to multiplication in the transform space. This guess is *wrong* as you can quickly see by looking at almost any example. For a concrete example, let $F(s) = \frac{1}{s}$ and $G(s) = \frac{1}{s^2}$ so that $f(t) = 1$ and $g(t) = t$ while $H(s) = F(s)G(s) = \frac{1}{s^3}$, so that $h(t) = t^2/2$. Thus $h(t) \neq f(t)g(t)$.

Let's continue with this example. Again suppose that $F(s) = \frac{1}{s}$ so that $f(t) = 1$, but assume now that $G(s) = \frac{n!}{s^{n+1}}$ so that $g(t) = t^n$. Now determine which function $h(t)$ has $F(s)G(s)$ as its Laplace transform:

$$h(t) = \mathcal{L}^{-1}\left\{F(s)G(s)\right\} = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+2}}\right\} = \frac{n!}{(n+1)!}t^{n+1} = \frac{1}{n+1}t^{n+1}.$$

What is the relationship between $f(t)$, $g(t)$, and $h(t)$? One thing that we can observe is that $h(t)$ is an integral of $g(t)$:

$$h(t) = \frac{1}{n+1}t^{n+1} = \int_0^t \tau^n \, d\tau = \int_0^t g(\tau)\, d\tau.$$

Let's try another example. Again let $F(s) = \frac{1}{s}$ so $f(t) = 1$, but now let $G(s) = \frac{s}{s^2+1}$ which implies that $g(t) = \cos t$. Then

$$h(t) = \mathcal{L}^{-1}\left\{F(s)G(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{s}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t,$$

and again we can observe is that $h(t)$ is an integral of $g(t)$:

$$h(t) = \sin t = \int_0^t \cos\tau \, d\tau = \int_0^t g(\tau)\, d\tau.$$

What these examples suggest is that *multiplication of* $G(s)$ *by* $\frac{1}{s} = \mathcal{L}\left\{1\right\}$ *in transform space corresponds to integration of* $g(t)$ *in the input space* $\mathcal{E}$. In fact, it is easy to see that this observation is legitimate by a calculation with the differentiation formula (1). Suppose that $G(s) \in \mathbb{R}_{\text{pr}}(s)$ is arbitrary and let $h(t) = \int_0^t g(\tau)\, d\tau$. Then $h'(t) = g(t)$ and $h(0) = 0$ so Equation (1) gives

$$G(s) = \mathcal{L}\left\{g(t)\right\} = \mathcal{L}\left\{h'(t)\right\} = sH(s) - h(0) = sH(s),$$

so that (in the language of Laplace transform pairs),

$$\int_0^t g(\tau)\,d\tau \longleftrightarrow H(s) = \frac{1}{s}G(s).$$

We will refer to this formula as division by s in transform space:

Division by s in transform space

$$\int_0^t f(\tau)\,d\tau \quad \longleftrightarrow \quad \frac{F(s)}{s}$$

We have thus determined the effect on the input space $\mathcal{E}$ of multiplying on the transform space by the Laplace transform of the function 1. Namely, the effect is integration. If we replace the function 1 by an arbitrary function $f(t) \in \mathcal{E}$, then the effect on $\mathcal{E}$ of multiplication by $F(s)$ is more complicated, but it can still be described by means of an integral operation. To describe this operation precisely, suppose that $f(t)$ and $g(t)$ are elementary functions. The **convolution product** or **convolution** of $f(t)$ and $g(t)$, is a new elementary function denoted by the symbol $f * g$, and defined by the integral formula

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{2}$$

What this formula means is that $f * g$ is the *name* of a new function constructed from $f(t)$ and $g(t)$ and the *value* of $f * g$ at the arbitrary point t is denoted by $(f * g)(t)$ and it is computed by means of the integral formula (2). Then the result we want is that the *convolution product of $f(t)$ and $g(t)$ on the input space $\mathcal{E}$ corresponds to the ordinary multiplication of $F(s)$ and $G(s)$ on the transform space $\mathbb{R}_{\mathrm{pr}}(s)$.* That is the content of the following theorem, the proof of which we will postpone until Chapter 4.

Theorem 2.5.1 (The Convolution Theorem). *Let $f(t)$, $g(t) \in \mathcal{E}$. Then*

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}. \tag{3}$$

In terms of inverse Laplace transforms, this is equivalent to the following statement. If $F(s)$ and $G(s)$ are in transform space then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}. \tag{4}$$

The following box summarizes the theorem in terms of transform pairs:

Convolution of input functions

$$(f * g)(t) \longleftrightarrow F(s)G(s)$$

An important special case of Equation (2) that is worth pointing out explicitly is

$$\boxed{(f * 1)(t) = \int_0^t f(\tau)\, d\tau.} \tag{5}$$

The convolution product $f * g$ behaves in many ways like an ordinary product:

$$\begin{aligned} f * g &= g * f && \text{(commutative property)} \\ (f * g) * h &= f * (g * h) && \text{(associative property)} \\ f * (g + h) &= f * g + f * h && \text{(distributive property)} \\ f * 0 &= 0 * f = 0 \end{aligned}$$

Indeed, these properties of convolution are easily verified from the definition (2). There is one significant difference, however. In general $f * 1 \neq f$. In fact, Equation (5) shows that $t * 1 = t^2/2 \neq t$. In other words, convolution by the constant function 1 does not behave like a multiplicative identity.

Example 2.5.2. Compute the convolution product $e^{at} * e^{bt}$ where $a \neq b$.

► **Solution.** Use the defining equation (2) to get

$$e^{at} * e^{bt} = \int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau = e^{bt} \int_0^t e^{(a-b)\tau}\, d\tau = \frac{e^{at} - e^{bt}}{a - b}.$$

Observe that

$$\mathcal{L}\left\{\frac{e^{at} - e^{bt}}{a - b}\right\} = \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right) = \frac{1}{(s-a)(s-b)} = \mathcal{L}\left\{e^{at}\right\}\mathcal{L}\left\{e^{bt}\right\},$$

so this calculation is in agreement with what is expected from Theorem 2.5.1. ◄

Example 2.5.3. Compute the convolution product $e^{at} * e^{at}$.

► **Solution.** Computing from the definition:

$$e^{at} * e^{at} = \int_0^t e^{a\tau} e^{a(t-\tau)}\, d\tau = e^{at} \int_0^\tau d\tau = te^{at}.$$

As with the previous example, note that the calculation

$$\mathcal{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2} = \mathcal{L}\left\{e^{at}\right\}\mathcal{L}\left\{e^{at}\right\}$$

agrees with the expectation of Theorem 2.5.1. ◄

Remark 2.5.4. Since

$$\lim_{a\to b} \frac{e^{at}-e^{bt}}{a-b} = \frac{d}{da}e^{at} = te^{at},$$

the previous two examples show that

$$\lim_{a\to b} e^{at} * e^{bt} = te^{at} = e^{at} * e^{at},$$

so that the convolution product is, in some sense, a continuous operation.

The convolution theorem is particularly useful in computing the inverse Laplace transform of a product.

Example 2.5.5. Compute the inverse Laplace transform of $\dfrac{s}{(s-1)(s^2+9)}$.

► **Solution.** The inverse Laplace transforms of $\dfrac{s}{s^2+9}$ and $\dfrac{1}{s-1}$ are $\cos 3t$ and e^t, respectively. The convolution theorem now gives

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s^2+9)}\right\} &= \cos 3t * e^t \\
&= \int_0^t \cos 3\tau\, e^{t-\tau}\, d\tau \\
&= e^t \int_0^t \cos 3\tau\, e^{-\tau}\, d\tau \\
&= \frac{e^t}{10}\left(-e^{-\tau}\cos 3\tau + 3e^{-\tau}\sin 3\tau\right)\Big|_0^t \\
&= \frac{1}{10}(-\cos 3t + 3\sin 3t + e^t)
\end{aligned}$$

◄

In the Table of Section C.3 a list is given of convolutions of some common functions. You may want to familiarize yourself with this table so as to know when you will be able to use it. The example above appears in the table ($a = 1$ and $b = 3$). Verify the answer.

Example 2.5.6. Compute the convolution product $t^m * t^n$ where $m, n \geq 0$.

▶ **Solution.** Start by computing

$$\mathcal{L}\left\{t^m * t^n\right\} = \mathcal{L}\left\{t^m\right\}\mathcal{L}\left\{t^n\right\} = \frac{m!}{s^{m+1}}\frac{n!}{s^{n+1}} = \frac{m!\,n!}{s^{m+n+2}}.$$

Now take the inverse Laplace transform to conclude

$$t^m * t^n = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{t^m * t^n\right\}\right\} = \mathcal{L}^{-1}\left\{\frac{m!\,n!}{s^{m+n+2}}\right\} = \frac{m!\,n!}{(m+n+1)!}t^{m+n+1}.$$

Thus

$$\boxed{t^m * t^n = \frac{m!\,n!}{(m+n+1)!}t^{m+n+1}.} \tag{6}$$

As special cases of this formula note that

$$t^2 * t^3 = \frac{1}{60}t^6 \quad \text{and} \quad t * t^4 = \frac{1}{30}t^6.$$

◀

Example 2.5.7. Find the inverse Laplace transform of $\dfrac{1}{s(s^2+1)}$.

▶ **Solution.** This could be done using partial fractions, but instead we will do the calculation using the division by s in transform space formula:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau\,d\tau = -\cos t + 1.$$

◀

Example 2.5.8. Consider the initial value problem

$$y'' + a^2y = f(t), \quad y(0) = 0, y'(0) = 0, \tag{7}$$

where $f(t) \in \mathcal{E}$ is an arbitrary elementary function. If we apply the Laplace transform $\mathcal{L}$ to this equation we obtain

$$(s^2 + a^2)Y(s) = F(s),$$

so that

$$Y(s) = \frac{1}{s^2 + a^2} F(s).$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + a^2)}\right\} = \frac{1}{a}\sin at,$$

the convolution theorem expresses $y(t)$ as a convolution product

$$y(t) = \frac{1}{a}\sin at * f(t).$$

This allows for the expression of $y(t)$ as an integral

$$y(t) = \frac{1}{a}\int_0^t f(\tau)\sin a(t - \tau)\, d\tau.$$

This integral equation can be thought of dynamically as starting from an arbitrary input function $f(t)$ and producing the output function $y(t)$ determined by the differential equation (7). Schematically,

$$f(t) \longmapsto y(t).$$

Moreover, although we arrived at this equation via the Laplace transform, it was never actually necessary to compute $F(s)$.

In the next example, we revisit a simple rational function whose inverse Laplace transform was computed by the techniques of Section 2.3 (see Equation (13) of that section).

Example 2.5.9. Compute the inverse Laplace transform of $\dfrac{1}{(s^2 + a^2)^2}$.

▶ **Solution.** The inverse Laplace transform of $1/(s^2 + a^2)$ is $(1/a)\sin at$. By the convolution theorem

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\} &= \frac{1}{a^2}\sin at * \sin at. \\
&= \frac{1}{a^2}\int_0^t \sin a\tau \sin a(t - \tau)\, d\tau \\
&= \frac{1}{a^2}\int_0^t \sin a\tau(\sin at\cos a\tau - \sin a\tau\cos at)\, d\tau \\
&= \frac{1}{a^2}\sin at\int_0^t \sin a\tau\cos a\tau\, d\tau - \cos at\int_0^t \sin^2 a\tau\, d\tau \\
&= \frac{1}{a^2}\left(\sin at\frac{\sin^2 at}{2a} - \cos at\frac{at - \sin at\cos at}{2a}\right) \\
&= \frac{1}{2a^3}(\sin at - at\cos at).
\end{aligned}$$

◀

Now, one should see how to handle $1/(s^2+a^2)^3$ and even higher powers: repeated applications of convolution. Let f^{*k} denote the convolution of f with itself k times. In other words

$$f^{*k} = f * f * \cdots * f, \qquad \text{k times.}$$

Then it is easy to see that

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+a^2)^n}\right) = \frac{1}{a^n}\sin^{*n} at$$

$$\text{and} \qquad \mathcal{L}^{-1}\left(\frac{s}{(s^2+a^2)^n}\right) = \frac{1}{a^{n-1}}\cos at * \sin^{*(n-1)} at.$$

There are explicit formulas for these convolutions. Although they are very complicated, for completeness of this text they are given below. The proofs are long and not included.

Proposition 2.5.10. *For the simple rational functions we have:*

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^{n+1}}\right\} &= \frac{2\sin at}{(2a)^{2n+1}}\left(\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{2n-2l}{n}\frac{(2at)^{2l}}{(2l)!}\right)\\
&\quad -\frac{2\cos at}{(2a)^{2n+1}}\left(\sum_{l=0}^{[\frac{n-1}{2}]}(-1)^l\binom{2n-2l-1}{n}\frac{(2at)^{2l+1}}{(2l+1)!}\right)\\
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^{n+1}}\right\} &= \frac{\sin at}{(2a)^{2n}}\left(\sum_{l=0}^{[\frac{n-1}{2}]}(-1)^l\binom{2n-2l-2}{n-1}\frac{2l+1}{n}\frac{(2at)^{2l+1}}{(2l+1)!}\right)\\
&\quad +\frac{\cos at}{(2a)^{2n}}\left(\sum_{l=1}^{[\frac{n}{2}]}(-1)^l\binom{2n-2l-1}{n-1}\frac{2l}{n}\frac{(2at)^{2l}}{(2l)!}\right)
\end{aligned}$$

Remark 2.5.11. Recall that any proper rational function $\dfrac{P(s)}{Q(s)}$ in the transform space $\mathbb{R}_{\text{pr}}(s)$ can be decomposed into a sum of partial fractions, and each partial fraction is a scalar multiple of one of the three simple rational functions:

$$\frac{1}{(s+r)^k}, \quad \frac{1}{(s^2+bs+c)^k} \quad \text{and} \quad \frac{s}{(s^2+bs+c)^k},$$

where r, b, c are some real numbers and $b^2-4c<0$, so that the quadratics are irreducible over $\mathbb{R}$. Since the inverse Laplace transform of the first of these functions is given by

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+r)^k}l\right\} = \frac{1}{(k-1)!}t^{k-1}e^{-rt},$$

it follows that the inverse Laplace transform of any proper rational function can be computed if one can handle the second and third types of simple rational functions listed above. For these types of simple rational functions, one must complete the square of the irreducible quadratic in the denominator and write them in the form

$$\frac{1}{((s-B)^2+C^2)^k} \quad \text{and} \quad \frac{s}{((s-B)^2+C^2)^k}.$$

Then use the translation principle (Equation (7) in Section 2.2) and the above Proposition to compute the inverse Laplace transform. Thus all such rational functions have, in principle, computable inverse Laplace transforms.

Exercises

Compute the convolution product of the following functions.

1. $t * t$
2. $t * t^3$
3. $3 * \sin t$
4. $(3t+1) * e^{4t}$
5. $\sin 2t * e^{3t}$
6. $(2t+1) * \cos 2t$
7. $t^2 * e^{-6t}$
8. $\cos t * \cos 2t$
9. $e^{2t} * e^{-4t}$
10. $t * t^n$
11. $e^{at} * \sin bt$
12. $e^{at} * \cos bt$

13. $\sin at * \sin bt$

14. $\sin at * \cos bt$

15. $\cos at * \cos bt$

Compute the Laplace transform of each of the following functions.

16. $f(t) = \int_0^t (t-\tau)\cos 2\tau \, d\tau$

► **Solution.** The key is to recognize the integral defining $f(t)$ as the convolution integral of two functions. Thus $f(t) = (\cos 2t) * t$ so that

$$F(s) = \mathcal{L}\{(\cos 2t) * t\} = \mathcal{L}\{\cos 2t\}\,\mathcal{L}\{t\} = \frac{s}{s^2+4}\frac{1}{s^2} = \frac{1}{s(s^2+4)}.$$

◄

17. $f(t) = \int_0^t (t-\tau)^2 \sin 2\tau \, d\tau$

18. $f(t) = \int_0^t (t-\tau)^3 e^{-3\tau} \, d\tau$

19. $f(t) = \int_0^t \tau^3 e^{-3(t-\tau)} \, d\tau$

20. $f(t) = \int_0^t \cos 5\tau \, e^{4(t-\tau)} \, d\tau$

21. $f(t) = \int_0^t \sin 2\tau \cos(t-\tau) \, d\tau$

22. $f(t) = \int_0^t \sin 2\tau \sin 2(t-\tau) \, d\tau$

In each of the following exercises compute the inverse Laplace transform of the given function by use of the convolution theorem.

23. $\dfrac{1}{(s-2)(s+4)}$

24. $\dfrac{1}{s^2-6s+5}$

25. $\dfrac{1}{(s^2+1)^2}$

26. $\dfrac{s}{(s^2+1)^2}$

27. $\dfrac{1}{(s+6)s^3}$

28. $\dfrac{2}{(s-3)(s^2+4)}$

29. $\dfrac{s}{(s-4)(s^2+1)}$

30. $\dfrac{1}{(s-a)(s-b)} \quad a \neq b$

31. $\dfrac{1}{s^2(s^2+a^2)}$

32. $\dfrac{G(s)}{s+2}$

33. $G(s)\dfrac{s}{s^2+2}$

Write the zero-state solution of each of the following differential equations in terms of a convolution integral involving the input function $f(t)$. You may wish to review Example 2.5.8 before proceeding.

34. $y'' + 3y = f(t)$

35. $y'' + 4y' + 4y = f(t)$

36. $y'' + 2y' + 5y = f(t)$

37. $y'' + 5y' + 6y = f(t)$

Chapter 3

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

The class of linear second order differential equations is of fundamental importance in the sciences. They arise naturally in describing mechanical and electrical systems, wave oscillations, and a variety of other problems. We introduced a few simple examples of second order differential equations in our discussion of the Laplace transform. In this chapter we give a more systematic presentation.

Before we get to the definitions and main theorems we illustrate how a second order differential equation arises from modelling a spring-body-dashpot system. This model may arise in a simplified version of a suspension system on a vehicle or a washing machine. Consider the three main objects in Figure 3.1: the spring, the body, and the dashpot (shock absorber). Our goal is to determine the motion of the body in such a

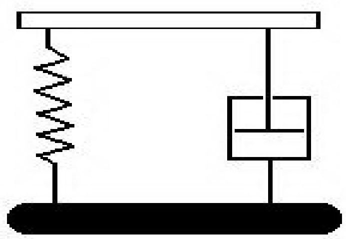


Figure 3.1: Spring-Body-Dashpot

system. Various forces come into play. These include the force of gravity, the restoring

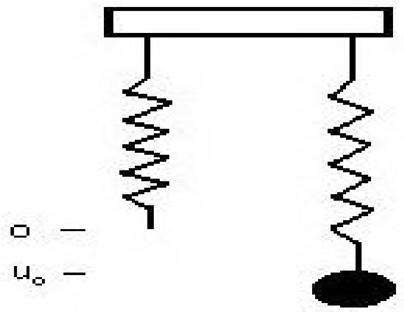


Figure 3.2: Spring-Body Equilibrium and Displacement

force of the spring, the damping force of the dashpot, and perhaps an external force. Let's examine these forces and how they are related. First, assume that the body has mass m. The force of gravity, F_G, acts on the body by the familiar formula

$$F_G = mg, \tag{1}$$

where g is the acceleration due to gravity. Our measurements will be positive in the downward direction so F_G is positive. When a spring is suspended with no mass attached the end of the spring will lie at a reference point ($u = 0$). Now, when a body is attached and allowed to come to equilibrium (i.e., no movement) it will stretch the spring a certain distance, u_0, say. This distance is called the **displacement** and is illustrated in Figure 3.2. The displacement is positive when the spring is stretched and negative when the spring is contracted. The force exerted by the spring to balance the force due to gravity is called the **restoring force**. It depends on the displacement and is denoted by $F_R(u_0)$. This balance gives us the equation

$$F_R(u_0) + F_G = 0. \tag{2}$$

Hooke's law says that the restoring force of many springs is proportional to the displacement, as long as the displacement is not too large. We will assume this. Thus, if u is the displacement we have

$$F_R(u) = -ku, \tag{3}$$

where k is a positive constant. When the displacement is positive (downward) the restoring force pulls the body upward hence the negative sign. Combining Equations (1), (2), and (3) gives us a formula for k,

$$k = \frac{mg}{u_0}.$$

In any practical situation there will be some kind of resistance to the motion of the body. In a suspension system there are shock absorbers. If our spring system were under water the viscosity of the water would dampen the motion (no pun intended) to a much greater extent than in air. In our system this resistance is represented by a dashpot and the force exerted by the dashpot is called the **damping force**, F_D. It depends on a lot of factors but an important factor is the velocity. To see that this is reasonable compare the difference in the forces against your head when you dive into a swimming pool off a 3 meter board and when you dive from the side of the pool. The greater the velocity when you enter the pool the greater your deceleration. We will assume that the damping force is proportional to the velocity. We thus have

$$F_D = -\mu v,$$

where $v = u'$ is velocity and μ is a positive constant known as the **damping constant**. The damping force acts in a direction opposite the velocity, hence the negative sign. We will let $F(t)$ denote an external force acting on the body. For example, this could be the varying forces acting on a suspension system due to driving over a bumpy road. If $a = u''$ is acceleration then Newton's second law of motion says that the total force of a body, given by mass times acceleration, is the sum of the forces acting on that body. We thus have

$$\text{Total Force} = F_G + F_R + F_D + \text{External Force},$$

which implies the equation

$$mu'' = mg - ku - \mu u' + F(t).$$

Recall from Equation 2 that $mg = -ku_0$. Substituting and combining terms gives

$$mu'' + \mu u' + k(u - u_0) = F(t).$$

If $y = u - u_0$ then y measures the displacement of the body from the spring-body equilibrium point, u_0. In this new variable we obtain

$$my'' + \mu y' + ky = F(t).$$

This is an example of a second order linear differential equation and the solutions that can be obtained vary dramatically depending on the constants m, k, and μ, and, of course, $F(t)$. Picture in your mind what happens in the following three situations where the external force is zero. In the first case, suppose the damping constant is 0. Then there is no friction. In this idealized system when the body is pulled from equilibrium and released it will oscillate up and down endlessly. In the second case, suppose the damping constant is very large. (Think of a vehicle with stiff shock absorbers.) When the body is pushed down and released it returns to rest without any oscillations. In the third case, suppose the damping constant is small, yet nonzero. Then when the body is pushed down and released it may oscillate several times but with decreasing heights until it comes to equilibrium. (In this case it's time to replace your shock absorbers.) A general discussion of the kinds of solutions one obtains is found in Section 3.7 where you will find graphs that represent the three situations described above. We will return to some specific examples of the spring-body-dashpot system in the last section, Section 3.8, where applications are considered.

For the next several sections we will study the mathematics of such second order differential equations. We can say a lot about the nature of the solution set and provide techniques for solving them.

3.1 Definitions and Conventions

Generally, a **second order differential equation** is an equation that involves a function, y say, its first derivative y', and its second derivative y''. Such an equation written in standard form looks like

$$y'' = F(t, y, y'), \tag{1}$$

and a **solution** is a function, $y(t)$, with at least two derivatives and satisfying (1). In other words,

$$y''(t) = F(t, y(t), y'(t)),$$

for all t in some interval I. A special case is the **linear** second order differential equation. They are the only type we will consider in this chapter.

A **linear second order differential equation** is an equation of the form

$$y'' + a(t)y' + b(t)y = f(t), \tag{2}$$

where a, b, and f are functions defined on some interval $I \subseteq \mathbb{R}$. The function f is called the **forcing function**. When $f = 0$, we call Equation (2) **homogeneous**, otherwise, it is called **nonhomogeneous**. If $a(t)$ and $b(t)$ are constant functions, then (2) is said to be **constant coefficient**. Note that a constant coefficient equation need not have the forcing function f a constant.

Example 3.1.1. Consider the following list of second order differential equations.

1. $y'' + y' + y = t$

2. $y'' + y' + ty = 1$
3. $y'' - 4y = 0$
4. $y'' - 4y = \sin 2t$
5. $t^2y'' + ty' + (t^2 - r)y = 0$
6. $3y'' + 2y' - 5y = 0$
7. $y'' - y'y$=0

Equations (1) and (4) are constant coefficient nonhomogeneous, Equation (2) is nonhomogeneous and is not constant coefficient, Equations (3) and (6) are homogeneous and constant coefficient, Equation (5) is homogeneous, but not constant coefficient, and Equation (7) is not even linear, so none of the adjectives homogeneous, nonhomogeneous, or constant coefficient apply.

The structure and nature of the set of solutions of linear differential equations is best understood in terms of linear operators. The left hand side of Equation (2) is made up of a combination of differentiation and multiplication by a function. Let $\boldsymbol{D}$ denote the derivative operator: $\boldsymbol{D}(y) = y'$. If $C^n(I)$ denotes the set of functions that have a continuous n^{th} derivative on the interval I then $\boldsymbol{D} : C^1(I) \to C^0(I)$. Note that we are using the convention that a 0^{th} derivative of g is just g itself, so that $C^0(I)$ is the set of continuous functions on the interval I. In general , $\boldsymbol{D} : C^{(n)}(I) \to C^{(n-1)}(I)$. In a similar way $\boldsymbol{D}^2$ will denote the second derivative operator. Thus $\boldsymbol{D}^2(y) = y''$ and $\boldsymbol{D}^2 : C^2(I) \to C^0(I)$. Let

$$\boldsymbol{L} = \boldsymbol{D}^2 + a\boldsymbol{D} + b, \tag{3}$$

where a and b are the same functions given in Equation (2). Thus $\boldsymbol{L}(y) = y'' + a(t)y' + b(t)y$ and Equation (2) can be rewritten $\boldsymbol{L}(y) = f$. We think of $\boldsymbol{L}$ as taking a function $y \in C^2(I)$ and producing a continuous function $\boldsymbol{L}(y) \in C^0(I)$.

Example 3.1.2. If $\boldsymbol{L} = \boldsymbol{D}^2 + 4t\boldsymbol{D} + 1$ then

- $\boldsymbol{L}(e^t) = (e^t)'' + 4t(e^t)' + 1(e^t) = (2 + 4t)e^t$
- $\boldsymbol{L}(\sin t) = -\sin t + 4t\cos t + \sin t = 4t\cos t$
- $\boldsymbol{L}(t^2) = 2 + 4t(2t) + (t^2) = 9t^2 + 2$
- $\boldsymbol{L}(t + 2) = 0 + 4t(1) + (t + 2) = 5t + 2$

The following proposition justifies calling $\boldsymbol{L}$ a **linear differential operator** (second order) and explains why Equation (2) is called a linear differential equation.

Proposition 3.1.3. *The operator $\boldsymbol{L}$ is linear. In other words, if f_1 and f_2 are in $C^2(I)$ and c_1 and c_2 are in $\mathbb{R}$ then*

$$\boldsymbol{L}(c_1 f_1 + c_2 f_2) = c_1 \boldsymbol{L}(f_1) + c_2 \boldsymbol{L}(f_2).$$

Proof. This follows from the fact that $\boldsymbol{D}$ and multiplication by a function are linear. □

The following theorem now gives the structure of the solution set to Equation (2).

Theorem 3.1.4. *Let $\boldsymbol{L}$ be a linear differential operator and f a function. Let $\mathcal{S}_{\boldsymbol{L}}^f$ be the solution set to the equation $\boldsymbol{L}(y) = f$ and $\mathcal{S}_{\boldsymbol{L}}^0$ the solution set to $\boldsymbol{L}(y) = 0$. Suppose $\varphi_p \in \mathcal{S}_{\boldsymbol{L}}^f$. Then $\mathcal{S}_{\boldsymbol{L}}^f = \varphi_p + \mathcal{S}_{\boldsymbol{L}}^0$. Furthermore, $\mathcal{S}_{\boldsymbol{L}}^0$ is a subspace. In other words, it is closed under addition and scalar multiplication.*

$$\boxed{\mathcal{S}_{\boldsymbol{L}}^f = \varphi_p + \mathcal{S}_{\boldsymbol{L}}^0}$$

Proof. Suppose φ_p is a fixed solution to $\boldsymbol{L}y = f$ and $\varphi_h \in \mathcal{S}_{\boldsymbol{L}}^0$. Then $\boldsymbol{L}(\varphi_p + \varphi_h) = f + 0 = f$ implies $\varphi_p + \varphi_h \in \mathcal{S}_{\boldsymbol{L}}^f$ by linearity of $\boldsymbol{L}$ (Proposition 3.1.3). On the other hand, if φ is some other solution to $\boldsymbol{L}y = f$ then again by linearity $\boldsymbol{L}(\varphi - \varphi_p) = f - f = 0$. Thus $\varphi - \varphi_p \in \mathcal{S}_{\boldsymbol{L}}^0$ and there is a function $\varphi_h \in \mathcal{S}_{\boldsymbol{L}}^0$ such that $\varphi = \varphi_p + \varphi_h$. This implies $\mathcal{S}_{\boldsymbol{L}}^f = \varphi_p + \mathcal{S}_{\boldsymbol{L}}^0$. Now suppose φ_1 and φ_2 are two homogeneous solutions and $a, b \in \mathbb{R}$. Then linearity implies $\boldsymbol{L}(a\varphi_1 + b\varphi_2) = a\boldsymbol{L}(\varphi_1) + b\boldsymbol{L}(\varphi_2) = 0 + 0 = 0$. This implies $\mathcal{S}_{\boldsymbol{L}}^0$ is closed under addition and scalar multiplication. □

Theorem 3.1.4 gives us a strategy for solving Equation (2): solve the homogeneous case first and then add on a particular solution.

Example 3.1.5. Determine the solution set to

$$y'' - y = t.$$

► **Solution.** In this example, the differential operator is $\boldsymbol{L} = \boldsymbol{D}^2 - 1$ and one is looking for solutions to $\boldsymbol{L}(\varphi(t)) = t$. It is easy to see that $\varphi(t) = -t$ is one such solution. Less obvious is the fact that the homogeneous equation

$$y'' - y = 0$$

has solutions $y = e^t$ and $y = e^{-t}$, but this could be determined, for example, by the Laplace transform techniques of Section 2.2. Since $\mathcal{S}_{\boldsymbol{L}}^0$ is a subspace the functions $y =$

$c_1e^t + c_2e^{-t}$ are also solutions. In fact, we will show that all solutions to $y'' - y = 0$ are of this form. Theorem 3.1.4 now implies that the solution set to $y'' - y = t$ is

$$\begin{aligned} \mathcal{S}_{\boldsymbol{L}}^t &= -t + \mathcal{S}_{\boldsymbol{L}}^0. \\ &= \left\{-t + c_1e^t + c_2e^{-t} : c_1, c_2 \in \mathbb{R}\right\} \end{aligned}$$

◀

The function φ is called a **particular solution** and $\mathcal{S}_{\boldsymbol{L}}^0$ is referred to as the **homogeneous solution set**. Of course, we will not leave it to guesswork to determine these. We will systematically deal with these questions, at least in the case of constant coefficient second order linear differential equations, in the next few sections. Nevertheless, we can already see from the above example that linearity is a very powerful property, which Theorem 3.1.4 exploits to describe the nature of the solution set for linear second order differential equations.

We can associate some **initial conditions** to Equation (2) of the form

$$y(t_0) = y_0 \quad y'(t_0) = y_1,$$

where $t_0 \in I$. Suppose in the above example that we included the initial conditions $y(0) = 2$ and $y'(0) = 0$. Then these conditions determine the constant c_1 and c_2 as follows:

$$\begin{aligned} 2 = y(0) = &\; 0 + c_1 + c_2 \\ 0 = y'(0) = &\; 0 + c_1 - c_2 \end{aligned}$$

This leads to a linear system of equations which, in this case, is very easy to solve. We obtain $c_1 = 1$ and $c_2 = 1$. The solution to the initial value problem then is

$$y = -t + e^t + e^{-t}.$$

This example illustrates the nature of what happens in general. The groundwork for this is laid next.

The Uniqueness and Existence Theorem

The following theorem is the fundamental theorem in this chapter. It guarantees that Equation (2) always has solutions if certain continuity conditions are assumed. Its proof, however, is beyond the scope of this book.

Theorem 3.1.6 (Uniqueness and Existence). *Suppose a, b, and f are continuous functions on an interval I. Let $t_0 \in I$. Then the initial value problem*

$$y'' + a(t)y' + b(t)y = f(t), \quad y(t_0) = y_0,\ y'(t_0) = y_1,$$

has a unique solution $\varphi(t)$, which is defined for all $t \in I$.

Theorem 3.1.6 does not tell us how to find any solution. We must develop procedures for this. Let's explain in more detail what this theorem does say. Under the conditions stated the Uniqueness and Existence theorem says that there always is a solution φ to the given initial value problem. The solution φ is at least twice differentiable on I and there is no other solution. In the preceding example we found $y = \varphi(t) = -t + e^t + e^{-t}$ a solution to $y'' - y = t$ with initial conditions $y(0) = 2$ and $y'(0) = 0$. Notice, in this case, that φ is, in fact, infinitely differentiable. The uniqueness part of Theorem 3.1.6 implies that there are no other solutions. In other words, there are no potentially hidden solutions, so that if we can find enough solutions to take care of all possible initial values, then Theorem 3.1.6 provides the theoretical underpinnings to know that we have found all possible solutions, and need look no further. Compare this theorem with the discussion in Section 1.5 where we saw examples (in the nonlinear case) of initial value problems which had infinitely many distinct solutions.

Let's consider another example.

Example 3.1.7. Find a solution to the following initial value problem:

$$y'' + y = t, \quad y(0) = 1,\ y'(0) = 0.$$

► **Solution.** We ask the student to verify the following assertions:

- $\varphi_p(t) = t$ is a solution to the differential equation
- $\sin t$ and $\cos t$ are homogeneous solutions.

Now $\varphi(t) = t + a\sin t + b\cos t$ is a solution for each $a, b \in \mathbb{R}$. The initial conditions imply

$$\begin{aligned} 1 &= y(0) = b \\ 0 &= y'(0) = 1 + a. \end{aligned}$$

Thus $a = -1$ and $b = 1$. Therefore, $\varphi(t) = t - \sin t + \cos t$ is a solution to the initial value problem. Theorem 3.1.6 implies there are no other solutions. ◄

Recall from Theorem 3.1.4 that once a particular solution is found the general solution is determined by the homogeneous case. Theorem 3.1.6 has much to say about the homogeneous case to which we turn our attention in the next section.

Exercises

For each of the following differential equations, determine if it is linear (yes/no). For each of those which is linear, further determine if the equation is homogeneous (homogeneous/nonhomogeneous) and constant coefficient (yes/no). Do **not** solve the equations.

1. $y'' + y'y = 0$

2. $y'' + y' + y = 0$

3. $y'' + y' + y = t^2$

4. $y'' + ty' + (1 + t^2)y^2 = 0$

5. $3y'' + 2y' + y = e^2$

6. $3y'' + 2y' + y = e^t$

7. $y'' + \sqrt{y'} + y = t$

8. $y'' + y' + y = \sqrt{t}$

9. $y'' - 2y = ty$

10. $y'' + 2y + t \sin y = 0$

11. $y'' + 2y' + (\sin t)y = 0$

12. $t^2y'' + ty' + (t^2 - 5)y = 0$

For each of the following linear differential operators $\boldsymbol{L}$ compute $\boldsymbol{L}(1)$, $\boldsymbol{L}(t)$, $\boldsymbol{L}(e^{-t})$, and $\boldsymbol{L}(\cos 2t)$. That is, evaluate $\boldsymbol{L}(y)$ for each of the given input functions.

13. $\boldsymbol{L}(y) = y'' + y$

Solution:: $\boldsymbol{L}(1) = 1'' + 1 = 1$; $\boldsymbol{L}(t) = t'' + t = t$; $\boldsymbol{L}(e^{-t}) = (e^{-t})'' + e^{-t} = 2e^{-t}$; and $\boldsymbol{L}(\cos 2t) = (\cos 2t)'' + \cos 2t = -4\cos 2t + \cos 2t = -3\cos 2t$.

14. $\boldsymbol{L}(y) = ty'' + y$

15. $\boldsymbol{L}(y) = 2y'' + y' - 3y$

16. $\boldsymbol{L} = \boldsymbol{D}^2 + 6\boldsymbol{D} + 5$

17. $\boldsymbol{L} = \boldsymbol{D}^2 - 4$

18. $\boldsymbol{L} = t^2\boldsymbol{D}^2 + t\boldsymbol{D} - 1$

19. If $\boldsymbol{L} = a\boldsymbol{D}^2 + b\boldsymbol{D} + c$ where a, b, c are real numbers, then show that $\boldsymbol{L}(e^{rt}) = (ar^2 + br + c)e^{rt}$. That is, the effect of applying the operator $\boldsymbol{L}$ to the exponential function e^{rt} is to multiply e^{rt} by the number $ar^2 + br + c$.

20. The differential equation $t^2y'' + ty' - y = t^{\frac{1}{2}}, \quad t > 0$ has a solution of the form $\varphi_p(t) = Ct^{\frac{1}{2}}$. Find C.

21. The differential equation $y'' + 3y' + 2y = t$ has a solution of the form $\varphi_p(t) = C_1 + C_2t$. Find C_1 and C_2.

22. Does the differential equation $y'' + 3y' + 2y = e^{-t}$ have a solution of the form $\varphi_p(t) = Ce^{-t}$? If so find C.

23. Does the differential equation $y'' + 3y' + 2y = e^{-t}$ have a solution of the form $\varphi_p(t) = Cte^{-t}$? If so find C.

24. Let $\boldsymbol{L}(y) = y'' + y$.

    (a) Check that $\varphi(t) = t^2 - 2$ is one solution to the differential equation $\boldsymbol{L}(y) = t^2$.

    (b) Check that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are two solutions to the differential equation $\boldsymbol{L}(y) = 0$.

    (c) Using the results of Parts (a) and (b), find a solution to each of the following initial value problems.

        i. $y'' + y = t^2, \quad y(0) = 1, \quad y'(0) = 0.$
        ii. $y'' + y = t^2, \quad y(0) = 0, \quad y'(0) = 1.$
        iii. $y'' + y = t^2, \quad y(0) = -1, \quad y'(0) = 3.$
        iv. $y'' + y = t^2, \quad y(0) = a, \quad y'(0) = b$, where $a, b \in \mathbb{R}$.

    Solution: Parts (a) and (b) are done by computing $y''+y$ where $y(t) = t^2-2$, $y(t) = \cos t$, or $y(t) = \sin t$. Then by Theorem 5.1.3, every function of the form $y(t) = t^2-2+c_1\cos t+$

$c_2 \sin t$ is a solution to $y'' + y = t^2$, where c_1 and c_2 are constants. If we want a solution to $\boldsymbol{L}(y) = t^2$ with $y(0) = a$ and $y'(0) = b$, then we need to solve for c_1 and c_2:

$$\begin{aligned} a &= y(0) = -2 + c_1 \\ b &= y'(0) = c_2 \end{aligned}$$

These equations give $c_1 = a + 2$, $c_2 = b$. Particular choices of a and b give the answers for i, ii, and iii.

25. Let $\boldsymbol{L}(y) = y'' - 5y' + 6y$.

    (a) Check that $\varphi(t) = \dfrac{1}{2}e^t$ is one solution to the differential equation $\boldsymbol{L}(y) = e^t$.

    (b) Check that $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ are two solutions to the differential equation $\boldsymbol{L}(y) = 0$.

    (c) Using the results of Parts (a) and (b), find a solution to each of the following initial value problems.

        i. $y'' - 5y' + 6y = e^t, \quad y(0) = 1, \quad y'(0) = 0.$
        ii. $y'' - 5y' + 6y = e^t, \quad y(0) = 0, \quad y'(0) = 1.$
        iii. $y'' - 5y' + 6y = e^t, \quad y(0) = -1, \quad y'(0) = 3.$
        iv. $y'' - 5y' + 6y = e^t, \quad y(0) = a, \quad y'(0) = b$, where $a, b \in \mathbb{R}$.

26. Let $\boldsymbol{L}(y) = t^2y'' - 4ty' + 6y$.

    (a) Check that $\varphi(t) = \dfrac{1}{6}t^5$ is one solution to the differential equation $\boldsymbol{L}(y) = t^5$.

    (b) Check that $y_1(t) = t^2$ and $y_2(t) = t^3$ are two solutions to the differential equation $\boldsymbol{L}(y) = 0$.

    (c) Using the results of Parts (a) and (b), find a solution to each of the following initial value problems.

        i. $t^2y'' - 4ty' + 6y = t^5, \quad y(1) = 1, \quad y'(1) = 0.$
        ii. $t^2y'' - 4ty' + 6y = t^5, \quad y(1) = 0, \quad y'(1) = 1.$
        iii. $t^2y'' - 4ty' + 6y = t^5, \quad y(1) = -1, \quad y'(1) = 3.$
        iv. $t^2y'' - 4ty' + 6y = t^5, \quad y(1) = a, \quad y'(1) = b$, where $a, b \in \mathbb{R}$.

For each of the following differential equations, find the largest interval on which a unique solution of the initial value problem

$$a_0(t)y'' + a_1(t)y' + a_3(t)y = f(t), \quad y(t_0) = y_1, \; y'(t_0) = y_1$$

is guaranteed by Theorem 5.2.1. Note that your interval may depend on the choice of t_0.

27. $t^2y'' + 3ty' - y = t^4$

    Solution: Write the equation in the standard form provided by Theorem 5.2.1:
    $$y'' + \frac{3}{t}y' - \frac{1}{t^2}y = t^2.$$
    Then $a(t) = \dfrac{3}{t}$, $b(t) = -\dfrac{1}{t^2}$, and $f(t) = t^2$. These three functions are all continuous on the intervals $(0, \infty)$ and $(-\infty, 0)$. Thus, Theorem 5.2.1 shows that if $t_0 \in (0, \infty)$ then the unique solution is also defined on the interval $(0, \infty)$, and if $t_0 \in (-\infty, 0)$, then the unique solution is defined on $(-\infty, 0)$.

28. $y'' - 2y' - 2y = \dfrac{1+t^2}{1-t^2}$

29. $(\sin t)y'' + y = \cos t$

30. $(1 + t^2)y'' - ty' + t^2y = \cos t$

31. $y'' + \sqrt{t}y' - \sqrt{t-3}y = 0$

32. $t(t^2 - 4)y'' + y = e^t$

33. The functions $y_1(t) = t^2$ and $y_2(t) = t^3$ are two distinct solutions of the initial value problem
    $$t^2y'' - 4ty' + 6y = 0, \quad y(0) = 0,\ y'(0) = 0.$$
    Why doesn't this violate the uniqueness part of Theorem 5.2.1?

34. Let $\varphi(t)$ be a solution of the differential equation
    $$y'' + a(t)y' + b(t)y = 0.$$
    We assume that $a(t)$ and $b(t)$ are continuous functions on an interval I, so that Theorem 5.2.1 implies that φ is defined on I. Show that if the graph of $\varphi(t)$ is tangent to the t-axis at some point t_0 of I, then $\varphi(t) = 0$ for all $t \in I$. *Hint:* If the graph of $\varphi(t)$ is tangent to the t-axis at $(t_0, 0)$, what does this say about $\varphi(t_0)$ and $\varphi'(t_0)$?

35. More generally, let $\varphi_1(t)$ and $\varphi_2(t)$ be two solutions of the differential equation
    $$y'' + a(t)y' + b(t)y = f(t),$$
    where, as usual we assume that $a(t)$, $b(t)$, and $f(t)$ are continuous functions on an interval I, so that Theorem 5.2.1 implies that φ_1 and φ_2 are defined on I. Show that if the graphs of $\varphi_1(t)$ and $\varphi_2(t)$ are tangent at some point t_0 of I, then $\varphi_1(t) = \varphi_2(t)$ for all $t \in I$.

3.2 The Homogeneous Case

In this section we are mainly concerned with the homogeneous case:

$$\boldsymbol{L}(y) = y'' + a(t)y' + b(t)y = 0 \tag{1}$$

The main result, Theorem 3.3.1 given below, shows that we will in principle be able to find two functions φ_1 and φ_2 such that all solutions to Equation (1) are of the form $c_1\varphi_1 + c_2\varphi_2$, for some constants c_1 and c_2.

Linear Independence

Two functions φ_1 and φ_2 defined on some interval I are said to be **linearly independent** if the equation

$$c_1\varphi_1 + c_2\varphi_2 = 0 \tag{2}$$

implies that c_1 and c_2 are both 0. Otherwise, we call φ_1 and φ_2 **linearly dependent**.

One must be careful about the meaning of this definition. We do not solve Equation (2). Rather, we are given that this equation is valid for all $t \in I$. With this information the focus is on what this says about the constants c_1 and c_2: are they necessarily both zero or not.

Let's consider two examples.

Example 3.2.1. First, let $\varphi_1(t) = t$ and $\varphi_2(t) = t^2$ be defined on $I = \mathbb{R}$. If the equation

$$c_1 t + c_2 t^2 = 0,$$

is valid for all $t \in \mathbb{R}$, then this implies, in particular, that

$$\begin{aligned} c_1 + c_2 &= 0 \qquad (\text{let } t = 1) \\ -c_1 + c_2 &= 0 \qquad (\text{let } t = -1) \end{aligned}$$

Now this system of linear equations is easy to solve. We obtain $c_1 = 0$ and $c_2 = 0$. Thus t and t^2 are linearly independent.

Example 3.2.2. In this second example let $\varphi_1(t) = t$ and $\varphi_2(t) = -2t$ defined on $I = \mathbb{R}$. Then there are many sets of constants c_1 and c_2 such that $c_1 t + c_2(-2t) = 0$. For example, we could choose $c_1 = 2$ and $c_2 = 1$. So the equation $c_1 t + c_2(-2t) = 0$ does *n*ot necessarily mean that c_1 and c_2 are zero. Hence t and $-2t$ are *n*ot independent. They are linearly dependent.

Remark 3.2.3. Notice that φ_1 and φ_2 are linearly dependent precisely when one function is a scalar multiple of the other, i.e., $\varphi_1 = \alpha\varphi_2$ or $\varphi_2 = \beta\varphi_1$ for $\alpha \in \mathbb{R}$ or $\beta \in \mathbb{R}$. In Example 3.2.1, $\varphi_2 \neq c\varphi_1$ while in Example 3.2.2, $\varphi_2 = -2\varphi_1$. Furthermore, given two linearly independent functions neither of them can be zero.

The main theorem for the homogeneous case

Theorem 3.2.4. *Let $\boldsymbol{L} = \boldsymbol{D}^2 + a\boldsymbol{D} + b$, where a, b are continuous functions on an interval I. Let $\mathcal{S}^0_{\boldsymbol{L}}$ be the solution set to $\boldsymbol{L}(y) = 0$. Then*

1. *There are two linearly independent solutions in $\mathcal{S}^0_{\boldsymbol{L}}$.*
2. *If $\varphi_1, \varphi_2 \in \mathcal{S}^0_{\boldsymbol{L}}$ are independent then any $\varphi \in \mathcal{S}^0_{\boldsymbol{L}}$ can be written $\varphi = c_1\varphi_1 + c_2\varphi_2$, for some $c_1, c_2 \in \mathbb{R}$.*

Proof. Let $t_0 \in I$. By Theorem 3.1.6, there are functions, ψ_1 and ψ_2, that are solutions to the initial value problems $\boldsymbol{L}(y) = 0$, with initial conditions $y(t_0) = 1,\ y'(t_0) = 0$ and $y(t_0) = 0,\ y'(t_0) = 1$, respectively. Suppose $c_1\psi_1 + c_2\psi_2 = 0$. Then

$$c_1\psi_1(t_0) + c_2\psi_2(t_0) = 0.$$

Since $\psi_1(t_0) = 1$ and $\psi_2(t_0) = 0$ it follows that $c_1 = 0$. Similarly we have,

$$c_1\psi_1'(t_0) + c_2\psi_2'(t_0) = 0.$$

Since $\psi_1'(t_0) = 0$ and $\psi_2'(t_0) = 1$ it follows that $c_2 = 0$. Therefore ψ_1 and ψ_2 are linearly independent. This proves (1).

Suppose $\varphi \in \mathcal{S}^0_{\boldsymbol{L}}$. Let $r = \varphi(t_0)$ and $s = \varphi'(t_0)$. Then $r\psi_1 + s\psi_2 \in \mathcal{S}^0_{\boldsymbol{L}}$ and

$$\begin{array}{rrcl} & r\psi_1(t_0) + s\psi_2(t_0) & = & \varphi(t_0) \\ \text{and} & r\psi_1'(t_0) + s\psi_2'(t_0) & = & \varphi'(t_0) \end{array}$$

This means the $r\psi_1 + s\psi_2$ and φ satisfy the same initial conditions. By the uniqueness part of Theorem 3.1.6 they are equal. Thus every solution is a linear combination of ψ_1 and ψ_2.

Now suppose φ_1 and φ_2 are any two linearly independent solutions in $\mathcal{S}_{\boldsymbol{L}}^0$ and suppose $\varphi \in \mathcal{S}_{\boldsymbol{L}}^0$. From the argument above we can write

$$\begin{aligned}\varphi_1 &= a\psi_1 + b\psi_2\\ \varphi_2 &= c\psi_1 + d\psi_2,\end{aligned}$$

which in matrix form can be written

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}.$$

We multiply both sides of this matrix equation by the adjoint $\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$ to obtain

$$\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}ad-bc & 0\\ 0 & ad-bc\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = (ad-bc)\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}.$$

Suppose $ad - bc = 0$. Then

$$\begin{aligned} d\varphi_1 - b\varphi_2 &= 0\\ \text{and}\quad -c\varphi_1 + a\varphi_2 &= 0.\end{aligned}$$

But since φ_1 and φ_2 are independent this implies that a, b, c, and d are zero which in turn implies that φ_1 and φ_2 are both zero. But this cannot be. We conclude that $ad-bc \neq 0$. We can now write ψ_1 and ψ_2 each as a linear combination of φ_1 and φ_2. Specifically,

$$\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}.$$

Since φ is a linear combination of ψ_1 and ψ_2 it follows the φ is a linear combination of φ_1 and φ_2. □

Remark 3.2.5. The matrix $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ that appears in the proof above appears in other contexts as well. For φ_1 and φ_2 in $\mathcal{S}_{\boldsymbol{L}}^0$ we define the **Wronskian matrix** by

$$W(\varphi_1, \varphi_2)(t) = \begin{bmatrix}\varphi_1(t) & \varphi_2(t)\\ \varphi_1'(t) & \varphi_2'(t)\end{bmatrix}$$

and the **Wronskian** by

$$w(\varphi_1, \varphi_2)(t) = \det W(\varphi_1, \varphi_2).$$

The relations in the proof

$$\begin{aligned}\varphi_1 &= a\psi_1 + b\psi_2\\ \varphi_2 &= c\psi_1 + d\psi_2\end{aligned}$$

when evaluated at t_0 imply that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \varphi_1(t_0) & \varphi_1'(t_0) \\ \varphi_1(t_0) & \varphi_2'(t_0) \end{bmatrix} = W(\varphi_1, \varphi_2)^t(t_0).$$

Since it was shown that $ad - bc \neq 0$ we have shown the following proposition.

Proposition 3.2.6. *Suppose φ_1 and φ_2 are linearly independent solutions in $\mathcal{S}_{\boldsymbol{L}}^0$. Then*

$$w(\varphi_1, \varphi_2) \neq 0.$$

On the other hand, given any two differentiable functions, φ_1 and φ_2, (not necessarily in $\mathcal{S}_{\boldsymbol{L}}^0$) whose Wronskian is a nonzero function then it is easy to see that φ_1 and φ_2 are independent. For suppose, t_0 is chosen so that $w(\varphi_1, \varphi_2)(t_0) \neq 0$ and $c_1\varphi_1 + c_2\varphi_2 = 0$. Then $c_1\varphi_1' + c_2\varphi_2' = 0$ and we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1\varphi_1(t_0) + c_2\varphi_2(t_0) \\ c_1\varphi_1'(t_0) + c_2\varphi_2'(t_0) \end{bmatrix} = W(\varphi_1, \varphi_2) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Simple matrix algebra[1] gives $c_1 = 0$ and $c_2 = 0$. Hence φ_1 and φ_2 are linearly independent.

Although one could check independence in this way it is simpler and more to the point to use the observation given in Remark 3.2.3.

Remark 3.2.7. Let's now summarize what Theorems 3.1.4, 3.1.6 and 3.2.4 tell us. In order to solve $\boldsymbol{L}(y) = f$ (satisfying the continuity hypotheses) we first need to find a particular solution φ_p, which exists by the Uniqueness and Existence Theorem 3.1.6. Next, Theorem 3.2.4 says that if φ_1 and φ_2 are any two linearly independent solutions of the associated homogeneous equation $\boldsymbol{L}(y) = 0$, then all of the solutions of the associated homogeneous equation are of the form $c_1\varphi_1 + c_2\varphi_2$. Theorem 3.1.4 now tells us that all solutions to $\boldsymbol{L}(y) = f$ are of the form $\varphi_p + c_1\varphi_1 + c_2\varphi_2$ for some choice of the constants c_1 and c_2. Furthermore, any set of initial conditions uniquely determine the constants c_1 and c_2.

A set $\{\varphi_1, \varphi_2\}$ of linearly independent solutions to the homogeneous equation $\boldsymbol{L}(y) = 0$ is called a **fundamental set** for the second order linear differential operator $\boldsymbol{L}$.

In the following sections we will develop methods for finding a fundamental set for $\boldsymbol{L}$ and a particular solution to the differential equation $\boldsymbol{L}(y) = f$. For now, let's illustrate the main theorems with a couple of examples.

[1]c.f. Chapter 5 for a discussion of matrices

Example 3.2.8. Let us reconsider the differential equation $y'' - y = -t$. In this case $\boldsymbol{L} = \boldsymbol{D}^2 - I$ and the forcing function is $f(t) = -t$. A particular solution is $\varphi_p(t) = t$. Two homogeneous solutions are $\varphi_1(t) = e^t$ and $\varphi_2(t) = e^{-t}$. They are independent since e^t and e^{-t} are not multiples of each other. Thus $\{e^t, e^{-t}\}$ forms a fundamental set for $\boldsymbol{L}(y) = 0$. By the above remark

$$\mathcal{S}_{\boldsymbol{L}}^f = \left\{t + c_1 e^t + c_2 e^{-t} : c_1, c_2 \in \mathbb{R}\right\}.$$

Example 3.2.9. Consider the differential equation $t^2 y'' + t y' + y = 2t$. In this case we divide by t^2 to rewrite the equation in standard form as $y'' + \frac{1}{t} y' + \frac{1}{t^2} y = \frac{2}{t}$ and observe that the coefficients are continuous on the interval $(0, \infty)$. Here $\boldsymbol{L} = \boldsymbol{D}^2 + \frac{1}{t}\boldsymbol{D} + \frac{1}{t^2}$ and the forcing function is $f(t) = \frac{1}{t^2}$. We leave the following verifications as an exercise:

1. A particular solution is $\varphi_p(t) = t$.
2. Two independent solutions of the homogeneous equation $\boldsymbol{L}(y) = 0$ are $\varphi_1(t) = \cos(\ln t)$ and $\varphi_2(t) = \sin(\ln t)$.

The set $\{\cos(\ln t), \sin(\ln t)\}$ is thus a fundamental set for $\boldsymbol{L}(y) = 0$. By the above remark the solution set to $\boldsymbol{L}(y) = f$ is given by

$$\mathcal{S}_{\boldsymbol{L}}^f = \{t + c_1 \cos(\ln t) + c_2 \sin(\ln t) : c_1, c_2 \in \mathbb{R}\}.$$

Exercises

Determine if each of the following pairs of functions are linearly independent or linearly dependent.

1. $\varphi_1(t) = 2t, \quad \varphi_2(t) = 5t$
2. $\varphi_1(t) = t^2, \quad \varphi_2(t) = t^5$
3. $\varphi_1(t) = e^{2t}, \quad \varphi_2(t) = e^{5t}$
4. $\varphi_1(t) = e^{2t+1}, \quad \varphi_2(t) = e^{2t-3}$

5. $\varphi_1(t) = \ln(2t), \quad \varphi_2(t) = \ln(5t)$

6. $\varphi_1(t) = \ln t^2, \quad \varphi_2(t) = \ln t^5$

7. $\varphi_1(t) = \sin 2t, \quad \varphi_2(t) = \sin t \cos t$

8. $\varphi_1(t) = \cosh t, \quad \varphi_2(t) = 3e^t(1 + e^{-2t})$

9. (a) Verify that $\varphi(t) = t^3$ and $\varphi_2(t) = |t^3|$ are linearly independent on $(-\infty, \infty)$.
   (b) Show that the Wronskian, $w(\varphi_1, \varphi_2)(t) = 0$ for all $t \in \mathbb{R}$.
   (c) Explain why Parts (a) and (b) do not contradict Theorem 3.2.6.
   (d) Verify that $\varphi_1(t)$ and $\varphi_2(t)$ are solutions to the linear differential equation

$$t^2 y'' - 2ty' = 0, \qquad , y(0) = 0, \quad y'(0) = 0.$$

   (e) Explain why Parts (a), (b), and (d) do not contradict Theorem 3.1.6.

3.3 Constant Coefficient Differential Operators

A **constant coefficient second order linear differential operator** has the form $\boldsymbol{L} = a\boldsymbol{D}^2 + b\boldsymbol{D} + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Throughout this section $\boldsymbol{L}$ will be of this form and we will determine explicitly the solution set of the homogeneous equation $\boldsymbol{L}(y) = ay'' + by' + cy = 0$. From Theorem 3.2.4 it is enough to find two linearly independent solutions.

Dividing this equation by a gives an equivalent equation in standard form. Since the coefficients b/a and c/a are constant and hence continuous on all of $\mathbb{R}$ any solution φ will exist as a function on all of $\mathbb{R}$. Therefore $\varphi(0)$ and $\varphi'(0)$ are defined. Let's then consider the Laplace transform of the equation:

$$ay'' + by' + c = 0, \quad y(0) = y_0, \ y'(0) = y_1. \tag{1}$$

Recall our convention: $Y = \mathcal{L}(y)$. We obtain

$$as^2 Y(s) - asy_0 - ay_1 + bsY(s) - by_0 + cY(s) = 0.$$

Solving for Y gives

$$Y(s) = \frac{ay_0 s + (by_0 + ay_1)}{as^2 + bs + c}. \tag{2}$$

The numerator of $Y(s)$ is a linear term and all possible linear terms are obtained by varying the initial conditions y_0 and y_1.

The polynomial $p(s) = as^2 + bs + c$ which appears in the denominator of $Y(s)$ is called the **characteristic polynomial** for $\boldsymbol{L}$. The characteristic polynomial also arises as the multiplier of the exponential e^{st} when the differential operator $\boldsymbol{L}$ is applied to e^{st}. That is,

$$\boldsymbol{L}(e^{st}) = p(s)e^{st}.$$

This equality is easily verified by direct substitution of e^{st} into $\boldsymbol{L}$. (See Exercise 19, Page 136.)

The partial fraction decomposition of $Y(s)$ is completely determined by the way $p(s)$ factors. Our experience with the Laplace transform tells us there are three possibilities to consider:

1. $p(s)$ has two distinct real roots
2. $p(s)$ has a repeated root
3. $p(s)$ has a pair of conjugate complex roots.

$p(s)$ has two distinct real roots

Suppose $p(s) = a(s - r_1)(s - r_2)$, where $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$. Then the partial fraction decomposition of $Y(s)$ has the form

$$Y(s) = \frac{A}{s - r_1} + \frac{B}{s - r_2}.$$

(The constant a can be absorbed into the constants A and B) The inverse Laplace transform is a linear combination of $e^{r_1 t}$ and $e^{r_2 t}$. Since they are not multiples of each other they are independent. The equations $L(e^{r_1 t}) = p(r_1)e^{r_1 t} = 0$ and $L(e^{r_2 t}) = p(r_2)e^{r_2 t} = 0$ imply that $\{e^{r_1 t}, e^{r_2 t}\}$ form a fundamental set to $\boldsymbol{L}(y) = 0$ by Theorem 3.2.4.

p has a repeated root

Here, we are supposing that $p(s) = a(s - r)^2$. In this case the partial fraction decomposition of $Y(s)$ has the form

$$Y(s) = \frac{A}{s - r} + \frac{B}{(s - r)^2}.$$

The inverse Laplace transform is a linear combination of e^{rt} and te^{rt}, which are independent. It is easy to check that both are solutions to $Ly = 0$ and hence $\{e^{rt}, te^{rt}\}$ is a fundamental set.

p has complex conjugate roots

In this case $p(s)$ does not factor over $\mathbb{R}$. Rather if the complex roots of $p(s)$ are $r = \alpha \pm i\beta$ with $\beta \neq 0$, then $p(s)$ can be rewritten in the form $p(s) = a(s-(\alpha+i\beta))(s-(\alpha-i\beta)) = a(s-\alpha)^2 + \beta^2$. In this case the partial fraction decomposition of $Y(s)$ is of the form

$$Y(s) = \frac{As + B}{(s-\alpha)^2 + \beta^2}.$$

The inverse Laplace transform is a linear combination of the functions $e^{\alpha t}\sin\beta t$ and $e^{\alpha t}\cos\beta t$. It is easy to see they are independent and solutions to $Ly = 0$. It follows from Theorem 3.2.4 that $e^{\alpha t}\sin\beta t$ and $e^{\alpha t}\cos\beta t$ form a fundamental set of solutions for the equation $\boldsymbol{L}(y) = 0$.

We now summarize our results as one theorem, which will be followed by several examples.

Theorem 3.3.1. *Suppose $\boldsymbol{L} = a\boldsymbol{D}^2 + b\boldsymbol{D} + c$ is a constant coefficient second order differential operator. Let $p(s) = as^2 + bs + c$ be the characteristic polynomial.*

1. *If r_1 and r_2 are two distinct real roots of $p(s)$ then*
$$\left\{e^{r_1 t},\, e^{r_2 t}\right\}$$
is a fundamental set for $\mathcal{S}^0_{\boldsymbol{L}}$.

2. *If r is a double root of $p(s)$ then*
$$\left\{e^{rt},\, te^{rt}\right\}$$
is a fundamental set for $\mathcal{S}^0_{\boldsymbol{L}}$.

3. *If $\alpha \pm i\beta$ are the complex conjugate roots of $p(s)$ then*
$$\left\{e^{\alpha t}\cos\beta t,\, e^{\alpha t}\sin\beta t\right\}$$
is a fundamental set for $\mathcal{S}^0_{\boldsymbol{L}}$.

Let's now consider some examples.

Example 3.3.2. Suppose $\boldsymbol{L} = 2\boldsymbol{D}^2 + 3\boldsymbol{D} + 1$. The characteristic polynomial is $p(s) = 2s^2 + 3s + 1 = (2s+1)(s+1)$. The roots are thus -1 and $-1/2$. By Theorem 3.3.1, $e^{-\frac{1}{2}t}$ and e^{-t} form a fundamental set for $\boldsymbol{L}(y) = 0$.

Example 3.3.3. Suppose $\boldsymbol{L} = \boldsymbol{D}^2 + -4\boldsymbol{D} + 4$. The characteristic polynomial is $p(s) = s^2 - 4s + 4 = (s-2)^2$. Thus 2 is a double root. By Theorem 3.3.1, e^{2t} and te^{2t} form a fundamental set for $\boldsymbol{L}(y) = 0$.

Example 3.3.4. Suppose $\boldsymbol{L} = \boldsymbol{D}^2 + 2\boldsymbol{D} + 3$. The characteristic polynomial is $p(s) = s^2 + 2s + 3 = (s+1)^2 + 2$. The roots are thus $-1 \pm i\sqrt{2}$. By Theorem 3.3.1, $e^{-t}\cos\sqrt{2}t$ and $e^{-t}\sin\sqrt{2}t$ form a fundamental set for $\boldsymbol{L}(y) = 0$.

It is worth emphasizing that once we have Theorem 3.3.1 we can write down the complete solution set of the homogeneous linear second order constant coefficient differential equation $\boldsymbol{L}(y) = 0$ directly from the knowledge of the algebraic roots of the characteristic polynomial $p(s)$. Thus, the Laplace transform has been used as a *tool* for deriving Theorem 3.3.1, and in the situations to which it applies, the theorem can be used directly, without going through the intermediate steps of calculating a Laplace transform and then an inverse Laplace transform. Of course, there are many situations where Theorem 3.3.1 does not apply, e.g., to nonhomogeneous equations, and for these additional techniques will be needed.

Exercises

Determine a fundamental set for each of the following differential equations. Use Examples 3.3.2 - 3.3.4 as guides.

1. $y'' + y' - 2y = 0$
2. $y'' - 16y = 0$
3. $y'' + 3y' = 0$
4. $2y'' - 5y' - 3y = 0$
5. $y'' - 2y = 0$
6. $y'' - 2y' - y = 0$
7. $y'' - 6y' + 9y = 0$
8. $y'' + 4y' + 4y = 0$

9. $y'' = 0$

10. $4y'' - 12y' + 9y = 0$

11. $y'' + y = 0$

12. $5y'' + y = 0$

13. $y'' - 4y' + 13y = 0$

14. $y'' + 2y' + 2y = 0$

15. $y'' - 8y' + 17y = 0$

16. $y'' + y' + y = 0$

Find the solution to the following initial value problems.

17. $y'' - y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = 1$

18. $y'' - 2y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$

19. $y'' + 4y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = 1$

20. $y'' + 4y = 0, \quad y(\pi) = 2, \quad y'(\pi) = -2$

21. $y'' - 7y = 0, \quad y(0) = 0, \quad y'(0) = 14$

22. $y'' + 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -1$

Find a second order linear homogeneous differential equation with constant real coefficients that has the given function as a solution, or explain why there is not such an equation.

23. $e^t + 2e^{-3t}$

24. $3e^{-2t} - 5e^{-7t}$

25. te^{-2t}

26. $\sin 5t$

27. $e^{2t} \sin 3t$

28. $\dfrac{2t}{e^t}$

29. $\dfrac{e^t}{2t}$

Verify that every solution to the following differential equations satisfies the limit condition

$$\lim_{t\to\infty} y(t) = 0.$$

30. $y'' + 5y' + 6y = 0$
31. $y'' + y' + y = 0$
32. $y'' + 2y' + 10y = 0$
33. Verify that some solutions of the differential equation

$$y'' - y' - 6y = 0$$

satisfy $\lim_{t\to\infty} y(t) = 0$, while others satisfy $\lim_{t\to\infty} y(t) = \pm\infty$.

3.4 The Cauchy-Euler Equations

When the coefficients of a second order linear differential operator are variable the corresponding equation can become very difficult to solve. Indeed, equations that might appear 'simple' may have no solution expressible in terms of common functions. New functions in fact frequently appear as solutions to differential equations which can not be expressed in terms of other known functions. For example the equation

$$t^2y'' + ty' + (t^2 - r)y = 0$$

where $r \in \mathbb{R}$ is an important differential equation that occurs in physical problems. The solutions cannot be expressed in terms of the standard elementary functions, i.e., polynomials, exponential, logarithm, and trig functions, but there nevertheless are solutions on $I = (0, \infty)$ by the Uniqueness and Existence theorem. These solutions, known as Bessel functions, have been thoroughly studied and one can find information about them in standard mathematical handbooks. We will not be studying this differential equation in this course, but there is a similar looking class of variable coefficient linear differential equations, known as Cauchy-Euler equations, for which the solutions are easy to obtain by techniques similar to those we have already learned for constant coefficient equations. We will consider these equations now.

A **Cauchy-Euler equation** is a second order linear differential equation of the following form:

$$at^2y'' + bty' + cy = 0, \tag{1}$$

where a, b and c are real constants and $a \neq 0$. When put in standard form we obtain:

$$y'' + \frac{b}{at}y' + \frac{c}{at^2}y = 0.$$

The functions $\frac{b}{at}$ and $\frac{c}{at^2}$ are continuous everywhere except at 0. Thus by the Uniqueness and Existence Theorem 3.1.6 solutions exist in either of the intervals $(-\infty, 0)$ or $(0, \infty)$. Of course, a solution need not, and in general, will not exist on the entire real line $\mathbb{R}$. To work in a specific interval we will assume $t > 0$. Let $\boldsymbol{L} = at^2\boldsymbol{D}^2 + bt\boldsymbol{D} + c$.

Laplace transform methods do not work in any simple fashion here. The change in variable $x = \ln t$ will transform Equation (1) into a constant coefficient differential equation. To see this observe that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx} \cdot \frac{1}{t}$$

so that

$$t\frac{dy}{dt} = \frac{dy}{dx}. \tag{2}$$

Similarly,

$$\begin{aligned} \frac{d^2y}{dt^2} &= \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}(\frac{1}{t}\frac{dy}{dx}) \\ &= \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dt}\frac{dy}{dx} \\ &= \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx} \end{aligned}$$

so that

$$t^2\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} - \frac{dy}{dx}. \tag{3}$$

Substituting Equations (2) and (3) into Equation (1) gives

$$a(\frac{d^2y}{dx^2} - \frac{dy}{dx}) + b\frac{dy}{dx} + cy = 0$$

or, equivalently, the linear constant coefficient equation,

$$a\frac{d^2y}{dx} + (b-a)\frac{dy}{dx} + cy = 0. \tag{4}$$

Let $q(s) = as^2 + (b-a)s + c$ be the characteristic polynomial. This polynomial is known as the **indicial polynomial** of the operator $\boldsymbol{L}$. As discussed in the previous section the way $q(s)$ factors determines the solutions to Equation (4) and thus the solutions to Equation (1). We consider the three possibilities.

q has distinct real roots

Suppose r_1 and r_2 are distinct roots to the indicial polynomial $q(s)$. Then $e^{r_1 x}$ and $e^{r_2 x}$ are solutions to Equation (4). Solutions to Equation (1) are obtained by the substitution $x = \ln t$: we have $e^{r_1 x} = e^{r_1 \ln t} = t^{r_1}$ and similarly $e^{r_2 x} = t^{r_2}$. Since t^{r_1} is not a multiple of t^{r_2} they are independent. By the main Theorem 3.2.4, $\{t^{r_1}, t^{r_2}\}$ is a fundamental set for $\boldsymbol{L}(y) = 0$.

3.4.1 q has a double root

Suppose r is a double root of q. Then e^{rx} and te^{rx} are independent solutions to Equation (4). The substitution $x = \ln t$ then gives t^r and $t^r \ln t$ as independent solutions to Equation (1). By Theorem 3.2.4 $\{t^r, t^r \ln t\}$ is a fundamental set for $\boldsymbol{L}(y) = 0$.

3.4.2 q has conjugate complex roots

Suppose q has complex roots $\alpha \pm i\beta$, where $\beta \neq 0$. Then $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ are independent solutions to Equation (4). The substitution $x = \ln t$ then gives $t^\alpha \cos(\beta \ln t)$ and $t^\alpha \sin(\beta \ln t)$ as independent solutions to Equation (1).

We now summarize the above results into one theorem.

Theorem 3.4.1. *Let $\boldsymbol{L} = at^2\boldsymbol{D}^2 + bt\boldsymbol{D} + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Let $q(s) = as^2 + (b-a)s + c$ be the indicial polynomial.*

1. *If r_1 and r_2 are distinct real roots of $q(s)$ then*
$$\{t^{r_1},\, t^{r_2}\}$$
is a fundamental set for $\boldsymbol{L}(y) = 0$.

2. *If r is a double root of $q(s)$ then*
$$\{t^r,\, t^r \ln t\}$$
is a fundamental set for $\boldsymbol{L}(y) = 0$.

3. *If $\alpha \pm i\beta$ are complex conjugate roots of $q(s)$, $\beta \neq 0$ then*
$$\{t^\alpha \sin(\beta \ln t),\, t^\alpha \cos(\beta \ln t)\}$$
is a fundamental set for $\boldsymbol{L}(y) = 0$.

Example 3.4.2. Consider the equation $t^2y'' - 2y = 0$. The indicial polynomial is $s^2 - s - 2 = (s-2)(s+1)$ and it has 2 and -1 as roots. Theorem 3.4.1 implies that $\{t^2, t^{-1}\}$ is a fundamental set for this Cauchy-Euler equation.

Example 3.4.3. The indicial polynomial for the Cauchy-Euler equation $4t^2y'' + 8ty' + y = 0$ is $4s^2 + 4s + 1 = (2s+1)^2$. Theorem 3.4.1 implies that $\left\{t^{-\frac{1}{2}}, t^{-\frac{1}{2}}\ln t\right\}$ is a fundamental set.

Example 3.4.4. Consider the equation $t^2y'' + ty' + y = 0$. The indicial polynomial is $s^2 + 1$ which has $\pm i$ as complex roots. Theorem 3.4.1 implies that $\{\cos\ln t, \sin\ln t\}$ is a fundamental set. This justifies item 2 in Example 3.2.9.

Exercises

Find the general solution of each of the following homogeneous Cauchy-Euler equations on the interval $(0, \infty)$.

1. $t^2y'' + 2ty' - 2y = 0$
2. $2t^2y'' - 5ty' + 3y = 0$
3. $t^2y'' + ty' - 2y = 0$
4. $4t^2y'' + y = 0$
5. $t^2y'' + 7ty' + 9y = 0$
6. $t^2y'' + ty' - 4y = 0$
7. $t^2y'' + ty' + 4y = 0$
8. $t^2y'' - ty' + 13y = 0$

Solve each of the following initial value problems.

9. $t^2y'' + 2ty' - 2y = 0, \quad y(1) = 0, \quad y'(1) = 1$
10. $4t^2y'' + y = 0, \quad y(1) = 2, \quad y'(1) = 0$

11. $t^2y'' + ty' + 4y = 0, \quad y(1) = -3, \quad y'(1) = 4$

12. $t^2y'' - 4ty' + 6y = 0, \quad y(0) = 1, \quad y'(0) = -1$

3.5 Undetermined Coefficients

In this section and the next we consider the nonhomogeneous differential equation

$$\boldsymbol{L}y = f, \tag{1}$$

where f is a nonzero function. The general theory developed in Sections 3.1 and 3.2, specifically Theorems 3.1.4 and 3.2.4, gives the strategy for solving Equation (1): First, we find the solution set, $\mathcal{S}_{\boldsymbol{L}}^0$, to the associated homogeneous equation $\boldsymbol{L}y = 0$. Second, we find a particular solution φ_p to Equation (1). Then the general solution takes the form

$$\varphi_p + \varphi_h,$$

where $\varphi_h \in \mathcal{S}_{\boldsymbol{L}}^0$. Previous sections have addressed the question of finding the solution set to a linear second order differential equation in some special circumstances. Thus our efforts now turn to finding a particular solution to Equation (1).

In this section we will describe a method, known as the **method of undetermined coefficients**, for finding a particular solution to

$$ay'' + by' + cy = f(t) \tag{2}$$

in the case where $f \in \mathcal{E}$ is an elementary function and a, b, and c are real numbers with $a \neq 0$. The general case will be considered in the next section.

Recall from Chapter 2 that an elementary function is a sum of constant multiples of the following types of simple elementary functions:

$$t^k e^{\alpha t}, \qquad t^k e^{\alpha t} \cos \beta t, \qquad t^k e^{\alpha t} \sin \beta t,$$

where α and β are real numbers and $k = 0, 1, 2, \ldots$. As we shall see any solution to Equation (2) is again an elementary function and therefore any particular solution φ_p can be expressed in the following way

$$\varphi_p = C_1\varphi_1 + \cdots + C_n\varphi_n, \tag{3}$$

where each φ_i, $i = 1, \dots n$, is a simple elementary function. If each of the φ_i's is not a solution to the associated homogeneous equation $ay'' + by' + cy = 0$ we will call Equation (3) the **form of the particular solution**. The method of undetermined coefficients allows one to determine the simple elementary functions that appear as terms in Equation (3). The coefficients of these term are then determined by substitution into Equation (2).

Let's consider the necessary details. Let $y(t)$ be the unique solution of Equation (2) subject to the initial conditions $y(0) = y_0$ and $y'(0) = y_1$, and, as usual, we let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to both sides of Equation (2) gives

$$a(s^2Y - sy_1 - y_1) + b(sY - y_0) + cY = F(s) = \frac{R(s)}{Q(s)},$$

and solving for Y we obtain

$$Y = \frac{y_0(as + b) + ay_1}{p(s)} + \frac{R(s)}{p(s)Q(s)}, \tag{4}$$

which we write as $Y_1(s) + Y_2(s)$, where $Y_1(s)$ is the first term and $Y_2(s)$ is the second term. Since Y is a proper rational function y is an elementary function. The first term in Equation (4), $Y_1(s)$, has inverse Laplace transform that is part of the solution to the associated homogeneous equation $ay'' + by' + cy = 0$. Since our focus is on finding the form of a particular solution we can ignore this contribution and concentrate on the second term $Y_2(s)$. The form of the partial fraction decomposition (see Page 96) of $Y_2(s)$ is completely determined by the factorization of the denominator $p(s)Q(s)$. The inverse Laplace transform of Y_2 is thus a sum of simple elementary functions. Some of these simple elementary functions may be included in φ_h and therefore ignored. It is the remaining terms that lead to the form of the particular solution. The way to proceed should become clear once we have illustrated the method with some simple examples.

Example 3.5.1. Find a particular solution $\varphi_p(t)$ to $y'' + 4y' - 5y = 3e^{-t}$.

► **Solution.** In this example, $f(t) = 3e^{-t}$ so that $F(s) = \frac{3}{s+1} = \frac{R(s)}{Q(s)}$ and $p(s) = s^2 + 4s - 5 = (s + 5)(s - 1)$; hence $p(s)Q(s) = (s + 5)(s - 1)(s + 1)$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{s+1},$$

and hence

$$y(t) = Ae^{-5t} + Be^{t} + Ce^{-t}.$$

The first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = Ce^{-t}$ where C is a constant, which can be determined by substitution into the original equation:

$$\begin{aligned}\varphi_p''(t) + 4\varphi_p'(t) - 5\varphi_p(t) &= Ce^{-t} - 4Ce^{-t} - 5Ce^{-t} \\ &= -8Ce^{-t} \\ &= 3e^{-t}.\end{aligned}$$

Thus, we must have $-8C = 3$ so that $C = -3/8$ and $\varphi_p(t) = (-3/8)e^{-t}$. The general solution to $y'' + 4y' - 5y = 3e^{-t}$ is then

$$y(t) = Ae^{-5t} + Be^{t} - \frac{3}{8}e^{-t},$$

where A and B are arbitrary real constants. ◀

Example 3.5.2. Find a particular solution $\varphi_p(t)$ to $y'' + 4y' - 5y = 3te^{-t}$.

▶ **Solution.** The only difference between this and the previous example is that now $f(t) = 3te^{-t}$ so that $F(s) = \dfrac{3}{(s+1)^2} = \dfrac{R(s)}{Q(s)}$. Hence $p(s)Q(s) = (s+5)(s-1)(s+1)^2$ which gives a partial fraction expansion for $Y(s)$ of the form

$$Y(s) = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2},$$

which then implies that

$$y(t) = Ae^{-5t} + Be^{t} + C_1e^{-t} + C_2te^{-t}.$$

As above, the first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = C_1e^{-t} + C_2te^{-t}$ where C_1 and C_2 are constants, which can be determined by substitution into the original equation, as follows. First compute the derivatives of $\varphi_p(t)$:

$$\begin{aligned}\varphi_p(t) &= C_1e^{-t} + C_2te^{-t} \\ \varphi_p'(t) &= (-C_1 + C_2)e^{-t} - C_2te^{-t} \\ \varphi_p''(t) &= (C_1 - 2C_2)e^{-t} + C_2te^{-t}.\end{aligned}$$

Now substitute these into the original equation:

$$\begin{aligned}\varphi_p''(t) + 4\varphi_p'(t) - 5\varphi_p(t) &= (-8C_1 + 2C_2)e^{-t} - 8C_2te^{-t} \\ &= 3te^{-t}.\end{aligned}$$

Setting $t = 0$ gives the equation $-8C_1 + 2C_2 = 0$, while comparing the coefficients of te^{-t} gives an equation $-8C_2 = 3$. Hence the coefficients C_1 and C_2 satisfy the system of equations

$$\begin{aligned} -8C_1 + 2C_2 &= 0 \\ -8C_2 = 3. \end{aligned}$$

Therefore, $C_2 = -3/8$ and $C_1 = C_2/4 = -3/32$, and we conclude that a particular solution $\varphi_p(t)$ to $y'' + 4y' - 5y = 3te^{-t}$ is given by

$$\varphi_p(t) = -\frac{3}{32}e^{-t} - \frac{3}{8}te^{-t}$$

and then general solution is

$$y(t) = \varphi_h(t) + \varphi_p(t) = Ae^{-5t} + Be^{t} - \frac{3}{32}e^{-t} - \frac{3}{8}te^{-t},$$

where A and B are arbitrary constants. ◀

Example 3.5.3. Find a particular solution $\varphi_p(t)$ to $y'' - 4y' - 5y = 3e^{-t}$.

▶ **Solution.** In this example, as in the first example, $f(t) = 3e^{-t}$ so that $F(s) = \dfrac{3}{s+1} = \dfrac{R(s)}{Q(s)}$. But $p(s) = s^2 - 4s - 5 = (s-5)(s+1)$; hence $p(s)Q(s) = (s-5)(s+1)^2$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A}{s-5} + \frac{B}{s+1} + \frac{C}{(s+1)^2},$$

and hence

$$y(t) = Ae^{5t} + Be^{-t} + Cte^{-t}.$$

The first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = Cte^{-t}$ where C is a constant, which can be determined by substitution into the original equation:

$$\begin{aligned} \varphi_p''(t) - 4\varphi_p'(t) - 5\varphi_p(t) &= C(-2+t)e^{-t} - 4C(1-t)e^{-t} - 5Cte^{-t} \\ &= -6Ce^{-t} \\ &= 3e^{-t}. \end{aligned}$$

Thus, we must have $-6C = 3$ so that $C = -1/2$ and $\varphi_p(t) = (-1/2)te^{-t}$. The general solution to $y'' - 4y' - 5y = 3e^{-t}$ is then

$$y(t) = Ae^{-5t} + Be^{t} - \frac{1}{2}te^{-t},$$

where A and B are arbitrary real constants. ◀

Example 3.5.4. Find a particular solution $\varphi_p(t)$ to $y'' + 2y' + y = 3e^{-t}$.

▶ **Solution.** Also in this example, $f(t) = 3e^{-t}$ so that $F(s) = \dfrac{3}{s+1} = \dfrac{R(s)}{Q(s)}$. But $p(s) = s^2 + 2s + 1 = (s+1)^2$; hence $p(s)Q(s) = (s+1)^3$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3},$$

and hence

$$y(t) = A_1 e^{-t} + A_2 t e^{-t} + (A_3/2) t^2 e^{-t}.$$

As in the previous examples, the first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = Ct^2 e^{-t}$ where C is a constant, which, as earlier, can be determined by substitution into the original equation:

$$\begin{aligned} \varphi_p''(t) + 2\varphi_p'(t) + \varphi_p(t) &= C(2 - 4t + t^2)e^{-t} + 2C(2t - t^2)e^{-t} + Ct^2 e^{-t} \\ &= 2Ce^{-t} \\ &= 3e^{-t}. \end{aligned}$$

Thus, $C = 3/2$ and $\varphi_p(t) = (3/2)t^2 e^{-t}$. The general solution to $y'' + 2y' + y = 3e^{-t}$ is then

$$y(t) = A_1 e^{-t} + A_2 t e^{-t} + \frac{3}{2} t^2 e^{-t},$$

where A_1 and A_2 are arbitrary real constants. ◀

Example 3.5.5. Find a particular solution $\varphi_p(t)$ to $y'' + 2y' + 5y = 3\sin 2t$.

▶ **Solution.** In this example, $f(t) = 3\sin 2t$ so that $F(s) = \dfrac{6}{s^2+4} = \dfrac{R(s)}{Q(s)}$, while $p(s) = s^2 + 2s + 5 = (s+1)^2 + 4$; hence $p(s)Q(s) = ((s+1)^2 + 4)(s^2 + 4)$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A_1 s + B_1}{(s+1)^2 + 4} + \frac{A_2 s + B_2}{s^2 + 4},$$

and hence (using Formulas 6) and 7) of Table C.2)

$$y(t) = \tilde{A}_1 e^{-t}\cos 2t + \tilde{B}_1 e^{-t}\sin 2t + A_2 \cos 2t + (B_2/2)\sin 2t.$$

As in the previous examples, the first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = C_1 \cos 2t + C_2 \sin 2t$ where C_1 and C_2 are constants to be determined by substitution into the original equation:

$$\begin{aligned}
\varphi_p''(t) + 2\varphi_p'(t) + 5\varphi_p(t) &= (-4C_1 + 4C_2 + 5C_1)\cos 2t + (-4C_2 - 4C_1 + 5C_2)\sin 2t \\
&= (C_1 + 4C_2)\cos 2t + (-4C_1 + C_2)\sin 2t \\
&= 3\sin 2t.
\end{aligned}$$

Thus, C_1 and C_2 satisfy the system of linear equations:

$$\begin{aligned}
C_1 + 4C_2 &= 0 \\
-4C_1 + C_2 &= 3.
\end{aligned}$$

Solving these equations for C_1 and C_2 gives

$$C_1 = -\frac{12}{17}, \quad C_2 = \frac{3}{17},$$

which implies that a particular solution $\varphi_p(t)$ to $y'' + 2y' + 5y = 3\sin 2t$ is

$$\varphi_p(t) = -\frac{12}{17}\cos 2t + \frac{3}{17}\sin 2t,$$

and the general solution is then

$$y(t) = A_1 e^{-t}\cos 2t + A_2 e^{-t}\sin 2t - \frac{12}{17}\cos 2t + \frac{3}{17}\sin 2t,$$

where A_1 and A_2 are arbitrary real constants. ◀

Example 3.5.6. Find a particular solution $\varphi_p(t)$ to $y'' + 4y = 3\sin 2t$.

▶ **Solution.** As in the previous example, $f(t) = 3\sin 2t$ so that $F(s) = \dfrac{6}{s^2+4} = \dfrac{R(s)}{Q(s)}$, but now $p(s) = s^2 + 4$; hence $p(s)Q(s) = (s^2+4)^2$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A_1 s + B_1}{s^2+4} + \frac{A_2 s + B_2}{(s^2+4)^2},$$

and hence (using Formulas from the Table of Convolutions)

$$y(t) = \tilde{A}_1 \cos 2t + \tilde{B}_1 \sin 2t + C_1 t\cos 2t + C_2 t\sin 2t.$$

As in the previous examples, the first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = C_1 t\cos 2t + C_2 t\sin 2t$ where C_1 and C_2 are constants to be determined by

Table 3.1: Form of a particular solution $\varphi_p(t)$

$p(s)$	$f(t)$	$Q(s)$	form of $\varphi_p(t)$
$(s+5)(s-1)$	$3e^{-t}$	$s+1$	Ce^{-t}
$(s+5)(s-1)$	$3te^{-t}$	$(s+1)^2$	$(C_1+C_2t)e^{-t}$
$(s-5)(s+1)$	$3e^{-t}$	$s+1$	Cte^{-t}
$(s+1)^2$	$3e^{-t}$	$s+1$	Ct^2e^{-t}
$(s+1)^2+4$	$3\sin 2t$	s^2+4	$C_1\cos 2t + C_2\sin 2t$
s^2+4	$3\sin 2t$	s^2+4	$C_1t\cos 2t + C_2t\sin 2t$

substitution into the original equation (the details of the substitution are left to the reader):

$$\begin{aligned}\varphi_p''(t) + 4\varphi_p(t) &= 4C_2\cos 2t - 4C_1\sin 2t \\ &= 3\sin 2t.\end{aligned}$$

Solving for C_1 and C_2 gives

$$C_1 = -\frac{3}{4}, \quad C_2 = 0,$$

which implies that a particular solution $\varphi_p(t)$ to $y'' + 4y = 3\sin 2t$ is

$$\varphi_p(t) = -\frac{3}{4}t\cos 2t,$$

and the general solution is then

$$y(t) = A_1\cos 2t + A_2\sin 2t - \frac{3}{4}\cos 2t,$$

where A_1 and A_2 are arbitrary real constants. ◀

We now summarize the calculations of the previous examples in Table 3.1. In this table, we are tabulating the *form* of a particular solution $\varphi_p(t)$ of the differential equation

$$ay'' + by' + cy = f(t)$$

as it relates to the characteristic polynomial $p(s) = as^2 + bs + c$, the forcing function $f(t)$, and the denominator $Q(s)$ of the Laplace transform $F(s) = R(s)/Q(s)$ of $f(t)$.

Notice that so long as $p(s)$ and $Q(s)$ do not have a common root (as in rows 1, 2, and 5 of Table 3.1), then the form of $\varphi_p(t)$ is exactly similar to that of $f(t)$, while if $p(s)$ and $Q(s)$ have a common root (either real or complex), then the form of $\varphi_p(t)$ is adjusted by multiplying by either t (if the common root is a simple root of $p(s)$, as in rows 3 and 6 of the table) or t^2 (if the common root is a double root of $p(s)$, as in row 4 of the table). These observations are formalized in the following theorem.

Theorem 3.5.7 (Undetermined Coefficients). *Let $p(s) = as^2 + bs + c$ be the characteristic polynomial of the nonhomogeneous constant coefficient differential equation*

$$ay'' + by' + cy = f(t).$$

The **form of a particular solution** *$\varphi_p(t)$ of this equation is determined by $f(t)$ and $p(s)$ in the following cases.*

1. *$f(t) = (A_0 + A_1 t + \cdots + A_k t^k)e^{\alpha t}$.*
   (a) *If $p(\alpha) \neq 0$, then the form of $\varphi_p(t)$ is*
   $$\varphi_p(t) = (C_0 + C_1 t + \cdots + C_k t^k)e^{\alpha t}.$$
   (b) *If $p(s) = a(s-\alpha)(s-r)$ with $r \neq \alpha$, i.e., α is a simple root of $p(s)$, then the form of $\varphi_p(t)$ is*
   $$\varphi_p(t) = t(C_0 + C_1 t + \cdots + C_k t^k)e^{\alpha t}.$$
   (c) *If $p(s) = a(s-\alpha)^2$, i.e., α is a double root of $p(s)$, then the form of $\varphi_p(t)$ is*
   $$\varphi_p(t) = t^2(C_0 + C_1 t + \cdots + C_k t^k)e^{\alpha t}.$$
2. *$f(t) = (A_0 + A_1 t + \cdots + A_k t^k)e^{\alpha t}\cos\beta t + (A'_0 + A'_1 t + \cdots + A'_k t^k)e^{\alpha t}\sin\beta t$.*
   (a) *If $p(\alpha + i\beta) \neq 0$ then the form of $\varphi_p(t)$ is*
   $$\varphi_p(t) = (C_0 + C_1 t + \cdots + C_k t^k)e^{\alpha t}\cos\beta t + (C'_0 + C'_1 t + \cdots + C'_k t^k)e^{\alpha t}\sin\beta t.$$
   (b) *If $p(\alpha + i\beta) = 0$ then the form of $\varphi_p(t)$ is*
   $$\varphi_p(t) = t(C_0 + C_1 t + \cdots + C_k t^k)e^{\alpha t}\cos\beta t + t(C'_0 + C'_1 t + \cdots + C'_k t^k)e^{\alpha t}\sin\beta t.$$

The form of $\varphi_p(t)$ means that C_1, C_2, ..., C_k, and C'_1, C'_2, ..., C'_k are initially undetermined coefficients which are computed by substitution into the differential equation.

Example 3.5.8. Determine the *form* of a particular solution $\varphi_p(t)$ for each of the following differential equations. Do not solve for the resulting constants.

1. $y'' - 5y' + 7y = 4e^{3t}$

   ▶ **Solution.** This is Case (1) with $\alpha = 3$. Since $p(s) = s^2 - 5s + 7$ and $p(3) = 1 \neq 0$, $\varphi_p(t) = Ce^{3t}$. ◀

2. $y'' - 5y' + 7y = 2t - t^3$

▶ **Solution.** This is Case (1) with $\alpha = 0$. Since $p(0) = 7 \neq 0$, it follows that

$$\varphi_p(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3.$$

◀

3. $y'' - 5y' = 2t - t^3$

▶ **Solution.** This is again Case (1) with $\alpha = 0$. Since $p(0) = 0$, it follows that

$$\varphi_p(t) = C_0 t + C_1 t^2 + C_2 t^3 + C_3 t^4.$$

◀

4. $y'' + y' - 6y = 2e^t - e^{2t}$

▶ **Solution.** Since $p(s) = (s-2)(s+3)$, we have that 2 is a simple root of $p(s)$. Hence, using both parts (a) and (b) of Case (1), it follows that $\varphi_p(t) = C_1 e^t + C_2 t e^{2t}$. ◀

5. $y'' + 4y = 5e^{-3t} \sin 2t$

▶ **Solution.** Since $\alpha + i\beta = -3 + 2i$ is not a root of $p(s) = s^2 + 4$, Case (2) shows

$$\varphi_p(t) = C_1 e^{-3t} \sin 2t + C_2 e^{-3t} \cos 2t.$$

◀

6. $y'' + 6y' + 13y = 5e^{-3t} \sin 2t$

▶ **Solution.** Since $\alpha + i\beta = -3 + 2i$ is a root of $p(s) = s^2 + 6s + 13 = (s+3)^2 + 4$, Case (2) shows

$$\varphi_p(t) = C_1 t e^{-3t} \sin 2t + C_2 t e^{-3t} \cos 2t.$$

◀

Exercises

Find the general solution of each of the following differential equations.

1. $y'' + 3y' + 2y = 4$
2. $y'' + 3y' + 2y = 12e^t$
3. $y'' + 3y' + 2y = \sin t$
4. $y'' + 3y' + 2y = \cos t$
5. $y'' + 3y' + 2y = 8 + 6e^t + 2\sin t$
6. $y'' - 3y' - 4y = 6e^t$
7. $y'' - 3y' - 4y = 5e^{4t}$
8. $y'' - 4y' + 3y = 20\cos t$
9. $y'' - 4y' + 3y = 2\cos t + 4\sin t$
10. $y'' - 4y = 8e^{2t} - 12$
11. $y'' - 3y' + 2y = 2t^3 - 9t^2 + 6t$
12. $y'' - 3y' + 2y = 2t^2 + 1$
13. $y'' + 4y = 5e^t - 4t$
14. $y'' + 4y = 5e^t - 4t^2$
15. $y'' + y' + y = t^2$
16. $y'' - 2y' - 8y = 9te^t + 10e^{-t}$
17. $y'' - 3y' = 2e^{2t}\sin t$
18. $y'' + y' = t^2 + 2t$
19. $y'' + y' = t + \sin 2t$
20. $y'' + y = \cos t$
21. $y'' + y = 4t\sin t$
22. $y'' - 3y' - 4y = 16t - 50\cos 2t$
23. $y'' + 4y' + 3y = 15e^{2t} + e^{-t}$
24. $y'' - y' - 2y = 6t + 6e^{-t}$

25. $y'' + y = \sin^2 t$ *Hint:* $\sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t$
26. $y'' - 4y' + 4y = e^{2t}$

Solve each of the following initial value problems.

27. $y'' - 5y' - 6y = e^{3t}$, $y(0) = 2$, $y'(0) = 1$.
28. $y'' + 2y' + 5y = 8e^{-t}$, $y(0) = 0$, $y'(0) = 8$.
29. $y'' + y = 10e^{2t}$, $y(0) = 0$, $y'(0) = 0$.
30. $y'' - 4y = 2 - 8t$, $y(0) = 0$, $y'(0) = 5$.
31. $y'' - y' - 2y = 5\sin t$, $y(0) = 1$, $y'(0) = -1$.
32. $y'' + 9y = 8\cos t$, $y(\pi/2) = -1$, $y'(\pi/2) = 1$.
33. $y'' - 5y' + 6y = e^t(2t - 3)$, $y(0) = 1$, $y'(0) = 3$.
34. $y'' - 3y' + 2y = e^{-t}$, $y(0) = 1$, $y'(0) = -1$.

3.6 Variation of Parameters

Let $\boldsymbol{L} = \boldsymbol{D}^2 + a\boldsymbol{D} + b$, where a and b are continuous functions on an interval I. In this section we address the issue of finding a particular solution to a nonhomogeneous linear differential equation $\boldsymbol{L}(y) = f$, where f is continuous on I. It is a pleasant and remarkable feature of linear differential equations that the homogeneous solutions can be used decisively to find a particular solution. The procedure we use is called **variation of parameters**.

Suppose $\{\varphi_1, \varphi_2\}$ is a fundamental set for $\boldsymbol{L}(y) = 0$. We know then that all solutions of the homogeneous equation $\boldsymbol{L}(y) = 0$ are of the form $c_1\varphi_1 + c_2\varphi_2$. To find a particular solution φ_p to $\boldsymbol{L}(y) = f$ the method of variation of parameters makes two assumptions. First, the parameters c_1 and c_2 are allowed to vary. We thus replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$, and assumes that the particular solution φ_p, takes the form

$$\varphi_p(t) = u_1(t)\varphi_1(t) + u_2(t)\varphi_2(t). \tag{1}$$

The second assumption is

$$u_1'(t)\varphi_1(t) + u_2'(t)\varphi_2(t) = 0. \tag{2}$$

What's remarkable is that these two assumptions consistently lead to explicit formulas for $u_1(t)$ and $u_2(t)$ and hence a formula for φ_p.

To simplify notation in the calculations that follow we will drop the 't' in expression like $u_1(t)$, etc. Before substituting φ_p into $\boldsymbol{L}(y) = f$ we first calculate φ_p' and φ_p''.

$$\begin{aligned} \varphi_p' = & \quad u_1'\varphi_1 + u_1\varphi_1' + u_2'\varphi_2 + u_2\varphi_2' \\ = & \quad u_1\varphi_1' + u_2\varphi_2' \\ & \quad \text{(by second assumption)}. \end{aligned}$$

Now for the second derivative

$$\varphi_p'' = \quad u_1'\varphi_1' + u_1\varphi_1'' + u_2'\varphi_2' + u_2\varphi_2''.$$

We now substitute φ_p into $\boldsymbol{L}(y)$.

$$\begin{aligned} \boldsymbol{L}(\varphi_p) &= \varphi_p'' + a\varphi_p' + b\varphi_p \\ &= u_1'\varphi_1' + u_1\varphi_1'' + u_2'\varphi_2' + u_2\varphi_2'' \\ &\qquad + a(u_1\varphi_1' + u_2\varphi_2') + b(u_1\varphi_1 + u_2\varphi_2) \\ &= u_1'\varphi_1' + u_2'\varphi_2' + u_1(\varphi_1'' + a\varphi_1' + b\varphi_1) + u_2(\varphi_2'' + a\varphi_2' + b\varphi_2) \\ &= u_1'\varphi_1' + u_2'\varphi_2' \\ &\quad \text{(because } \varphi_1 \text{ and } \varphi_2 \text{ are homogeneous solutions)} \end{aligned}$$

The second assumption and the equation $\boldsymbol{L}(\varphi_p) = f$ now lead to the following system:

$$\begin{aligned} u_1'\varphi_1 + u_2'\varphi_2 &= 0 \\ u_1'\varphi_1' + u_2'\varphi_2' &= f \end{aligned}$$

which can be rewritten in matrix form as

$$\begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}. \qquad (3)$$

The left most matrix in Equation (3) is none other than the Wronskian matrix, $W(\varphi_1, \varphi_2)$, which has a nonzero determinant because $\{\varphi_1, \varphi_2\}$ is a fundamental set (cf Theorem 3.2.4 and Proposition 3.2.6). By Cramer's rule, we can solve for u_1' and u_2'. We obtain

$$\begin{aligned} u_1' &= \frac{-\varphi_2 f}{w(\varphi_1, \varphi_2)} \\ u_2' &= \frac{\varphi_1 f}{w(\varphi_1, \varphi_2)}. \end{aligned}$$

We now obtain an explicit formula for a particular solution:

$$\begin{aligned} \varphi_p(t) &= u_1\varphi_1 + u_2\varphi_2 \\ &= \Big(\int \frac{-\varphi_2 f}{w(\varphi_1,\varphi_2)}\Big)\varphi_1 + \Big(\int \frac{\varphi_1 f}{w(\varphi_1,\varphi_2)}\Big)\varphi_2. \end{aligned} \tag{4}$$

The following theorem consolidates these results with Theorems 3.1.4 and 3.2.4.

Theorem 3.6.1. *Let $\boldsymbol{L} = \boldsymbol{D}^2 + a\boldsymbol{D} + b$, where a and b are continuous on an interval I. Suppose $\{\varphi_1, \varphi_2\}$ is a fundamental set of solutions for $\boldsymbol{L}(y) = 0$. If f is continuous on I then a particular solution, φ_p, to $\boldsymbol{L}(y) = f$ is given by the formula*

$$\varphi_p = \Big(\int \frac{-\varphi_2 f}{w(\varphi_1,\varphi_2)}\Big)\varphi_1 + \Big(\int \frac{\varphi_1 f}{w(\varphi_1,\varphi_2)}\Big)\varphi_2.$$

Furthermore, the solution set $\mathcal{S}_L^f$ to $\boldsymbol{L}(y) = f$ becomes

$$\mathcal{S}_L^f = \{\varphi_p + c_1\varphi_1 + c_2\varphi_2 : c_1, c_2 \in \mathbb{R}\}.$$

Remark 3.6.2. Equation (4), which gives an explicit formula for a particular solution, is too complicated to memorize and we do not recommend students to do this. Rather the point of variation of parameters is the method that leads to Equation (4) and our recommended starting point is Equation (3). You will see such matrix equations as we proceed in the text.

We will illustrate the method of variation of parameters with two examples.

Example 3.6.3. Consider the linear differential equation

$$y'' - y = -t.$$

We considered this equation earlier and noticed that $\varphi_p = t$ was a solution. We will use variation of parameters to derive this. Let $\boldsymbol{L} = \boldsymbol{D}^2 - 1$. The characteristic polynomial is

$$p(s) = s^2 - 1 = (s+1)(s-1),$$

which has -1 and 1 as roots. By Theorem 3.3.1 $\{e^{-t}, e^t\}$ is a fundamental set. The matrix equation

$$\begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

leads to the system

$$\begin{array}{rcrcl} e^{-t}u_1' & + & e^t u_2' & = & 0 \\ -e^{-t}u_1' & + & e^t u_2' & = & t. \end{array}$$

Adding these equations together gives $2e^t u_2' = t$ and hence

$$u_2' = \frac{t}{2}e^{-t}.$$

Subtracting the bottom equation from the top gives $2e^{-t}u_1' = -t$ and hence

$$u_1' = -\frac{t}{2}e^{t}.$$

Integration by parts then gives

$$u_1 = \frac{1}{2}(te^t - e^t)$$

and

$$u_2 = \frac{1}{2}(te^{-t} + e^{-t}).$$

We now substitute u_1 and u_2 into Equation (1) and obtain

$$\varphi_p(t) = \frac{1}{2}(te^t - e^t)e^{-t} + \frac{1}{2}(te^{-t} + e^{-t})e^t = t.$$

Theorem 3.1.4 implies

$$\mathcal{S}_{\boldsymbol{L}}^f = \left\{t + c_1e^{-t} + c_2e^t : u_1, c_2 \in \mathbb{R}\right\}.$$

Example 3.6.4. Let's consider the following equation:

$$t^2y'' - 2y = t^2 \ln t.$$

In standard form this becomes

$$y'' - \frac{2}{t^2}y = \ln t.$$

The associated homogeneous equation is $y'' - \frac{2}{t^2}y = 0$ and is a Cauchy-Euler equation. The indicial polynomial is $q(s) = s^2 - s - 2 = (s-2)(s+1)$, which has 2 and -1 as roots. Thus $\{t^{-1}, t^2\}$ is a fundamental set to the homogeneous equation $y'' - \frac{2}{t^2}y = 0$, by Theorem 3.4.1. The matrix equation

$$\begin{bmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \ln t \end{bmatrix}$$

leads to the system

$$\begin{array}{rcrcl} t^{-1}u_1' & + & t^2u_2' & = & 0 \\ -t^{-2}u_1' & + & 2tu_2' & = & \ln t. \end{array}$$

Multiplying the bottom equation by t and then adding the equations together gives $3t^2u_2' = t\ln t$ and hence

$$u_2' = \frac{1}{3t}\ln t.$$

Substituting u_2' into the first equation and solving for u_1' gives

$$u_1' = -\frac{t^2}{3}\ln t.$$

Integration by parts leads to

$$u_1 = -\frac{1}{3}(\frac{t^3}{3}\ln t - \frac{t^3}{9})$$

and a simple substitution leads to

$$u_2 = \frac{1}{6}(\ln t)^2.$$

We substitute u_1 and u_2 into Equation (1) to get

$$\varphi_p(t) = -\frac{1}{3}(\frac{t^3}{3}\ln t - \frac{t^3}{9})t^{-1} + \frac{1}{6}(\ln t)^2 t^2 = \frac{t^2}{54}(9(\ln t)^2 - 6\ln t + 2).$$

It follows from Theorem 3.1.4 that the solution set is

$$\left\{\frac{t^2}{54}(9(\ln t)^2 - 6\ln t + 2) + c_1 t^{-1} + c_2 t^2 : c_1, c_2 \in \mathbb{R}\right\}.$$

Exercises

Solve the following differential equations. Examples 3.6.3 and 3.6.4 will be helpful guides.

1. $y'' + y = \tan t$
2. $y'' + y = \sin t$
3. $y'' - 4y = e^{2t}$
4. $y'' - 2y' + y = \dfrac{e^t}{t}$
5. $y'' - 3y' + 2y = e^{3t}$
6. $y'' - 2y' + 5y = e^t$
7. $y'' + y = \sec t$
8. $y'' + 3y' = e^{-3t}$

9. $t^2y'' - 2ty' + 2y = t^4$

10. The differential equation $ty'' - y' = 3t^2 - 1$ has homogeneous solutions $\varphi_1(t) = 1$ and $\varphi_2(t) = t^2$. Find the general solution.

11. Show that the constants of integration in the formula for φ_p in Theorem 3.6.1 can be chosen so that a particular solution can be written in the form:

$$\varphi_p = \int_0^t \frac{\left|\begin{bmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1(t) & \varphi_2(t) \end{bmatrix}\right|}{\left|\begin{bmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_2'(x) & \varphi_2'(x) \end{bmatrix}\right|} f(x)dx$$

For each problem below use the result of problem 11 to obtain a particular solution to the given differential equation in the form given. Solve the differential equation using the Laplace transform method and compare.

12. $y'' + a^2y = f(t) \quad y_p(t) = \frac{1}{a}f(t) * \sin at$

13. $y'' - a^2y = f(t) \quad y_p(t) = \frac{1}{a}f(t) * \sinh at$

14. $y'' - 2ay' + a^2y = f(t) \quad y_p(t) = \frac{1}{a}f(t) * te^{-at}$

15. $y'' - (a+b)y' + aby = f(t),\ a \neq b \quad y_p(t) = \frac{1}{b-a}f(t) * (e^{bt} - e^{at})$

3.7 Harmonic Motion

A number of important applications of differential equations involve the type of equation studied in the previous sections. Two distinct types of physical problems which both employ the same mathematical model are problems involving the spring-body-dashpot systems, as discussed in the introduction to this chapter, and certain electric circuits. Both of these types of systems are modeled by means of a differential equation

$$ay'' + by' + cy = f(t) \tag{1}$$

where $y(t)$ represents displacement from equilibrium in a mass spring system or $y(t)$ represents the charge (or current) in an electric circuit, while a, b, and c are *positive* real constants, and $f(t)$ is a forcing function (or applied voltage in the case of an electric circuit). We will leave the analysis of the physical significance of the constants a, b, c to some applications that appear in the next section and more advanced courses in science and engineering. We shall instead concentrate on the mathematical problem of extracting information about the solutions of Equation (1). For your information, we will simply record what each of the terms in Equation (1) means in the two manifestations mentioned, namely, a spring system, and an electric circuit.

Table 3.2: Constants in Applied Problems

Equation Part	**Spring System**	**Electric Circuit**
y	Displacement	Charge Q
y'	Velocity	Current I
a	Mass	Inductance L
b	Damping Constant	Resistance R
c	Spring Constant	$(\text{Capacity})^{-1}$ $1/C$
$f(t)$	Applied Force	Applied Voltage $E(t)$

We will break our analysis of Equation (1) into several parts: free motion ($f(t) \equiv 0$) and forced motion ($f(t) \not\equiv 0$) and each of these is divided into undamped ($b = 0$) and damped ($b \neq 0$) motion.

Undamped Free Motion

In this case Equation (1) becomes

$$ay'' + cy = 0, \tag{2}$$

with $a > 0$ and $c > 0$. The characteristic polynomial of this equation is $p(s) = as^2 + c$ which has roots $\pm i\beta$ where $\beta := \sqrt{\frac{c}{a}}$, and hence Equation (2) has the general solution

$$y = c_1 \cos \beta t + c_2 \sin \beta t. \tag{3}$$

Using the trigonometric identity $\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$, Equation (3) can be rewritten as

$$y = A\cos(\beta t - \delta) \tag{4}$$

where $A = \sqrt{c_1^2 + c_2^2}$ and δ is obtained from the pair of equations $c_1 = A\cos\delta$ and $c_2 = A\sin\delta$, i.e., $\tan\delta = \frac{c_2}{c_1}$. Therefore, the graph of $y(t)$ satisfying Equation (2) is a pure cosine function with frequency β and with period

$$T = \frac{2\pi}{\beta} = 2\pi\sqrt{\frac{a}{c}}.$$

The numbers A and δ are commonly referred to as the *amplitude* and *phase angle* of the system. From Equation (4) we see that C is the maximum possible value of the function $y(t)$, and that $|y(t)| = A$ precisely when $t = \dfrac{\delta + n\pi}{\beta}$ where $n \in \mathbb{Z}$. This motion is illustrated in Figure 3.3.

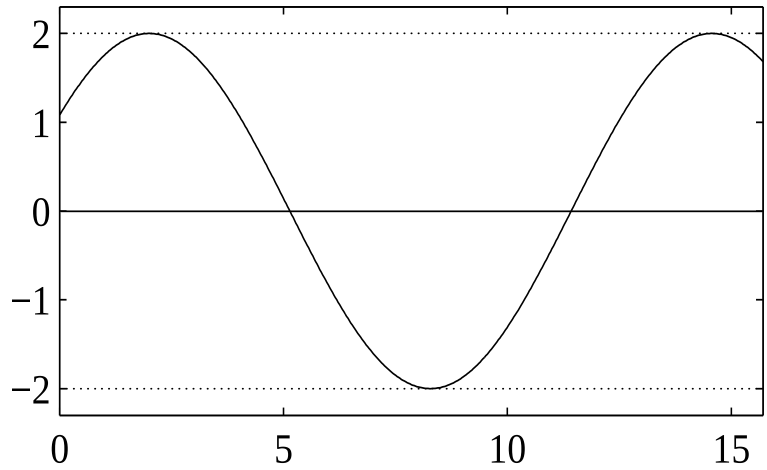


Figure 3.3: Undamped harmonic motion: Graph of $y(t) = 2\cos(.5t - 1)$

Example 3.7.1. The initial value problem

$$y'' + 3y = 0, \quad y(0) = -3, \quad y'(0) = 3$$

is easily seen to have the solution $y = -3\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t$ which can be rewritten in the form of Equation (4) by computing the amplitude $A = \sqrt{(-3)^2 + (\sqrt{3})^2} = 2\sqrt{3}$ and for the phase angle δ, we have $\tan\delta = \dfrac{\sqrt{3}}{-3} = -\dfrac{1}{\sqrt{3}}$. Hence $\delta = -\dfrac{\pi}{6}$, and thus

$$y = 2\sqrt{3}\cos\left(\sqrt{3}t + \frac{\pi}{6}\right).$$

Damped Free Motion

In this case we include the damping term by' by assuming that $b > 0$, which will, in fact, be the case in applications since the coefficient b represents the presence of friction (or resistance in an electrical circuit), and friction can never be completely eliminated. Thus we want solutions to the equation

$$ay'' + by' + cy = 0 \tag{5}$$

where we assume that $a > 0$, $b > 0$, and $c > 0$. In this case the characteristic polynomial $p(s) = as^2 + bs + c$ has roots r_1 and r_2 given by the quadratic formula

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{6}$$

and the nature of the solutions of Equation (5) are determined by the discriminant $D = b^2 - 4ac$ of the characteristic polynomial $p(s)$.

I. $D > 0$. In this case the two roots r_1 and r_2 in Equation (6) are distinct real roots so the general solution of Equation (5) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{7}$$

Moreover, note that both r_1 and r_2 are *negative* real numbers.

II. $D = 0$. In this case the characteristic polynomial $p(s)$ has only one root, namely $r = -\dfrac{b}{2a}$, and this root is negative since a and b are positive. Then the general solution of Equation (5) is

$$y = c_1 e^{rt} + c_2 t e^{rt} = (c_1 + c_2 t)e^{rt}. \tag{8}$$

III. $D < 0$. In this case the roots of the characteristic polynomial $p(s)$ are a pair of conjugate complex numbers $\alpha \pm i\beta$ where $\alpha = -\dfrac{b}{2a} < 0$ and $\beta = \dfrac{\sqrt{-D}}{2a} = \dfrac{\sqrt{4ac - b^2}}{2a}$. Then the general solution of Equation (5) is

$$y = e^{\alpha t}\left(c_1 \cos \beta t + c_2 \sin \beta t\right). \tag{9}$$

Notice that in all three cases, no matter what the constants c_1 and c_2, it will follow that

$$\lim_{t \to \infty} y(t) = 0.$$

Thus the motion $y(t)$ dies out as t increases. In case I, we say the motion is *overdamped*, in case II, the motion is said to be *critically damped*, and in case III, the motion is said to be *underdamped*. In the case of overdamped and critically damped motion, one can show (see Exercise 5.7.6) that $y(t) = 0$ for at most one value of t. The graphs of these cases are illustrated in Figure 3.4.

In the case of underdamped motion, Equation (4) shows that Equation (9) can be rewritten in the form

$$y = Ae^{\alpha t} \cos\left(\beta t - \delta\right) \tag{10}$$

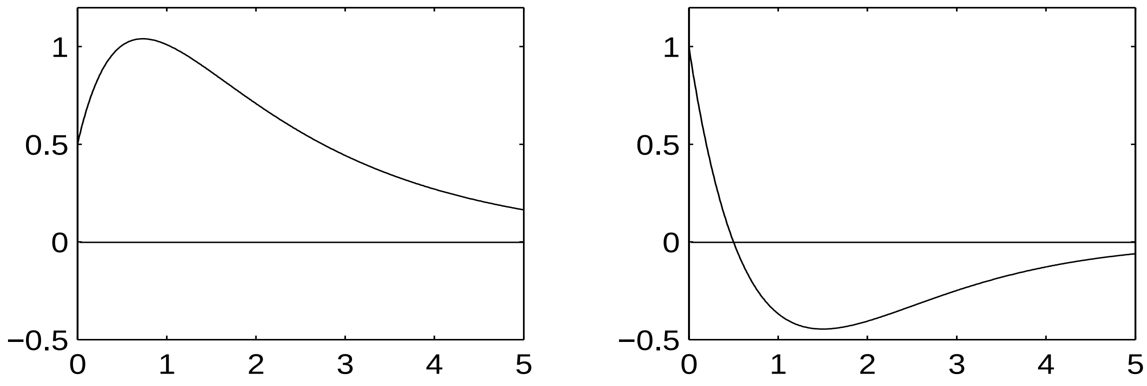


Figure 3.4: Overdamped and critically damped harmonic motion

where, as earlier, $A = \sqrt{c_1^2 + c_2^2}$ and $\tan\delta = \dfrac{c_2}{c_1}$. The graph is illustrated in Figure 3.5. Notice that y appears to be a cosine curve in which the amplitude decreases with time, and as in the case of undamped motion, one sees that $y(t) = 0$ for $t = \dfrac{\delta + \frac{(2n+1)\pi}{2}}{\beta}$ where $n \in \mathbb{Z}$.

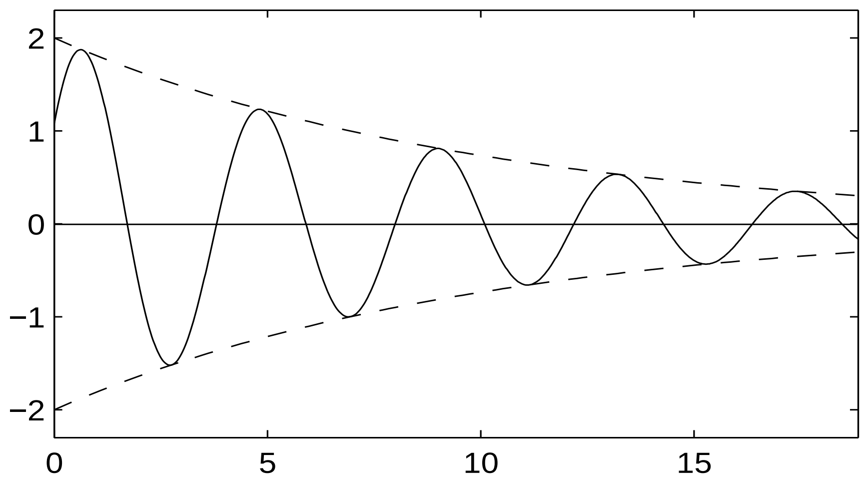


Figure 3.5: Underdamped harmonic motion: Graph of $y(t) = 2e^{-0.1t}\cos(1.5t - 1)$.

Undamped Forced Motion

Undamped forced motion refers to a system governed by a differential equation

$$ay'' + cy = f(t),$$

where $f(t)$ is a nonzero forcing function. We will only consider the special case where the forcing function is given by $f(t) = F_0 \cos \omega t$ where F_0 is a nonzero constant. Thus we are interested in describing the solutions of the differential equation

$$ay'' + cy = F_0 \cos \omega t \tag{11}$$

where, as usual, $a > 0$ and $c > 0$. From Equation (3) we know that a general solution to $ay'' + cy = 0$ is $y_h = c_1 \cos \beta t + c_2 \sin \beta t$ where $\beta = \sqrt{\frac{c}{a}}$, so if we can find a single solution $\varphi_p(t)$ to Equation (11), then Theorem 3.1.4 shows that the entire solution set is given by

$$\mathcal{S} = \{\varphi_p(t) + c_1 \cos \beta t + c_2 \sin \beta t : c_1,\ c_2 \in \mathbb{R}\}.$$

To find $\varphi_p(t)$ we shall solve Equation (11) subject to the initial conditions $y(0) = 0$, $y'(0) = 0$. As usual, if $Y = \mathcal{L}(y)(s)$, then we apply the Laplace transform to Equation (11) and solve for $Y(s)$ to get

$$Y(s) = \frac{1}{as^2 + c}\frac{F_0 s}{s^2 + \omega^2} = \frac{F_0}{a\beta}\frac{\beta}{s^2 + \beta^2}\frac{s}{s^2 + \omega^2}. \tag{12}$$

Then the convolution theorem (Theorem 2.5.1) shows that

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{F_0}{a\beta} \sin \beta t * \cos \omega t. \tag{13}$$

The following convolution formula comes from Table C.3:

$$\sin \beta t * \cos \omega t = \begin{cases} \dfrac{\beta}{\beta^2 - \omega^2}(\cos \omega t - \cos \beta t) & \text{if } \beta \neq \omega \\ \\ \dfrac{1}{2} t \sin \omega t & \text{if } \beta = \omega. \end{cases} \tag{14}$$

Combining Equations (13) and (14) gives

$$y(t) = \begin{cases} \dfrac{F_0}{a(\beta^2 - \omega^2)}(\cos \omega t - \cos \beta t) & \text{if } \beta \neq \omega \\ \\ \dfrac{F_0}{2a\omega} t \sin \omega t & \text{if } \beta = \omega. \end{cases} \tag{15}$$

We will first consider the case $\beta \neq \omega$ in Equation (15). Notice that, in this case, the solution $y(t)$ is the sum of two cosine functions with equal amplitude $(= \frac{F_0}{a(\beta^2-\omega^2)})$, but different frequencies β and ω. Recall the trigonometric identity

$$\cos(\theta-\varphi) - \cos(\theta+\varphi) = 2\sin\theta\sin\varphi.$$

If we set $\theta-\varphi = \omega t$ and $\theta+\varphi = \beta t$ and solve for $\theta = \frac{(\beta+\omega)t}{2}$ and $\varphi = \frac{(\beta-\omega)t}{2}$, we see that we can rewrite the first part of Equation (15) in the form

$$y(t) = \frac{2F_0}{a(\beta^2-\omega^2)} \sin\frac{(\beta-\omega)t}{2} \sin\frac{(\beta+\omega)t}{2}. \tag{16}$$

One may think of the function $y(t)$ as a sine function, namely $\sin\frac{(\beta+\omega)t}{2}$ (with frequency $\frac{\beta+\omega}{2}$) which is multiplied by another function, namely $\frac{2F_0}{a(\beta^2-\omega^2)}\sin\frac{(\beta-\omega)t}{2}$ which functions as a time varying amplitude function. The interesting case is when β is close to ω so that $\beta+\omega$ is close to 2ω and $\beta-\omega$ is close to 0. In this situation, one sine function changes very rapidly, while the other, which represents the change in amplitude, changes very slowly. See Figure 3.6. This type of phenomenon, known as *beats*, can be heard when one tries to tune a piano. When the frequency of vibration of the string is close to that of the tuning fork, one hears a pulsating beat which disappears when the two frequencies coincide.

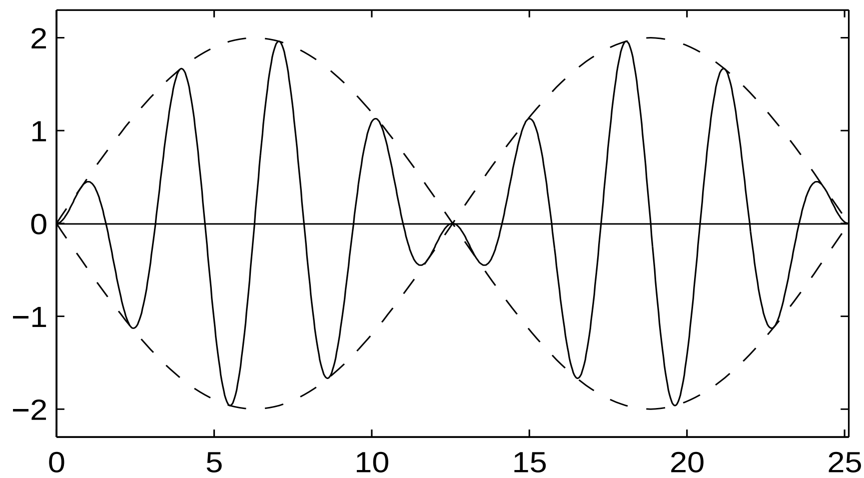


Figure 3.6: Beats: Graph of $y(t) = 2\sin(.25t)\sin 2t$.

In the case $\beta = \omega$ in Equation (15), the solution

$$y(t) = \frac{F_0}{2a\omega} t \sin\omega t$$

is unbounded as $t \to \infty$ and thus cannot represent the actual situation present in a physical system. Nevertheless it is useful as an idealized representation of what happens to a vibrating system if a force is applied to a vibrating system at a frequency ω close to that of the natural frequency β of the system. The resulting amplification of vibration can become large enough to destroy a mechanical or electrical system. The phenomenon of natural and applied frequencies being equal is known as *resonance*. This phenomenon can be used in a positive way to tune a radio to a particular frequency.

Damped Forced Motion

As in the previous section we will only consider forcing functions of the form $f(t) = F_0 \cos \omega t$ where F_0 is a constant. Thus we are interested in analyzing the solutions of the equation

$$ay'' + by' + cy = F_0 \cos \omega t \tag{17}$$

where a, b, c and F_0 are positive constants. It is a straightforward (albeit tedious) calculation to check that the function

$$\varphi_p(t) = \frac{F_0}{(c-\omega^2 a)^2 + b^2\omega^2}\big((c-\omega^2 a)\cos \omega t + b\omega \sin \omega t\big)$$

is a solution of Equation (17). Using Equation (4), this can be rewritten as

$$\varphi_p(t) = \frac{F_0}{\sqrt{(c-\omega^2 a)^2 + b^2\omega^2}} \cos(\omega t - \delta) \tag{18}$$

where $\tan \delta = \dfrac{b\omega}{c - \omega^2 a}$. Combining this with Equation (9), the general solution to Equation (17) is

$$y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) + \frac{F_0}{\sqrt{(c-\omega^2 a)^2 + b^2\omega^2}} \cos(\omega t - \delta) \tag{19}$$

where $\alpha = -\dfrac{b}{2a} < 0$. Notice that this implies that $\lim_{t\to\infty}(y(t) - \varphi_p(t)) = 0$, which says that every general solution of Equation (17) converges asymptotically to the particular solution $\varphi_p(t)$. For this reason, the solution $\varphi_p(t)$ is usually referred to as the *steady state solution* to the equation, while the solution $y(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$ of the associated homogeneous equation is referred to as a *transient solution*.

Exercises

Write the solution of each of the following initial value problems in the form $y(t) = A\cos(\beta t - \delta)$. See Example 5.7.1 for the method.

1. $y'' + 25y = 0, \quad y(0) = -2, \quad y'(0) = 10$

2. $y'' + 4y = 0, \quad y(0) = 3,\ y'(0) = -8$

3. $\frac{1}{2}y'' + 8y = 0, \quad y(0) = 1, \quad y'(0) = 2$

4. $y'' + y = 0, \quad y(0) = -1,\ y'(0) = -\sqrt{3}$

For each of the following differential equations, determine if the equation is underdamped, critically damped, or overdamped.

5. $y'' + y' + y = 0$

6. $y'' + 2y' + y = 0$

7. $y'' + 3y' + y = 0$

8. $2y'' + 5y' + y = 0$

9. $5y'' + 2y' + y = 0$

10. $4y'' + 4y' + y = 0$

Write each of the following functions in the form $y(t) = Ae^{\alpha t}\cos(\beta t - \delta)$.

11. $y(t) = e^{-t}(\cos 2t - \sin 2t)$

12. $y(t) = e^{-2t}(\sin t + \sqrt{3}\cos t)$

13. $y(t) = e^{-0.2t}(4\cos 5t - 3\sin 5t)$

14. For each of the functions in the previous exercise, find the smallest $t > 0$ with $y(t) = 0$.

15. Suppose that $y(t)$ is the solution to the initial value problem $ay'' + cy = 0$, $y(0) = y_0$, $y'(0) = y_1$ and we will assume that $a > 0$, $c > 0$ so that the equation is that of undamped harmonic motion. Verify that the amplitude of the motion is
$$A = \sqrt{y_0^2 + \frac{ay_1^2}{c}}.$$

16. Let $ay'' + by' + cy = 0$ be an equation of damped harmonic motion. If the motion is critically damped or overdamped, verify that any solution $y(t)$ can have $y(t) = 0$ for *at most* one value of t. *Hint:* Look carefully at Equations (5.7.7) and (5.7.8).

    Express each of the following functions in the form $A \sin \alpha t \sin \beta t$.

17. $\cos 9t - \cos 7t$

18. $\cos 9t - \cos 10t$

3.8 Applications

In this section we return to the spring-body-dashpot system we considered in the introduction. We will look as some numerical examples and study them in the light of the previous section. It may be helpful to review the introduction for concepts and terminology. In each of the examples given below it will be assumed that springs obey Hooke's law and the damping force is proportional to velocity.

Units of Measurement

There are two systems of measurements that are commonly used in examples like these: The English and Metric systems. The following table summarizes the units.

System	**Time**	**Distance**	**Mass**	**Force**
Metric	seconds (s)	meters (m)	kilograms (kg)	Newtons (N)
English	seconds (s)	feet (ft)	slugs (sl)	pounds (lbs)

The next table summarizes quantities derived from these units.

Quantity	**Formula**
velocity (v)	distance / time
acceleration (a)	distance / time^2
force (F)	mass $\cdot$ acceleration
spring constant (k)	force / distance
damping constant (μ)	force $\cdot$ time / distance

In the metric system one Newton of force (N) will accelerate a one kilogram mass (kg) one m/s^2. In the English system a one pound force (lb) will accelerate a one slug mass (sl) one ft/s^2. To compute the mass of a body in the English system one must divide the weight by the acceleration due to gravity, which is $g = 32$ ft/sec^2 near the earths surface. Thus a body weighing 64 lbs is 2 slugs. To compute the gravitational force in the metric system one must multiply the mass by the acceleration due to gravity, which is $g = 9.8$ m/sec^2. Thus a 5 kg mass exerts a gravitational force of 47.5 N.

Examples of Spring-Body-Dashpot Systems

In the examples below keep the following formulas in mind:

Gravitational force	$F_G = mg$	where g is acceleration due to gravity
Restoring force	$F_R = -ku_0$	where k is the spring constant and u_0 is spring displacement
Damping force	$F_D = -\mu v$	where μ is the damping constant and $v = y'$ is velocity
External force	F	

The initial value problem for the spring-body-dashpot problem is

$$my'' + \mu y' + ky = F(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

where y_0 is the initial position of the body and y_1 is initial velocity.

Example 3.8.1. Suppose a spring is stretched 8 inches from its natural length when a body weighing 4 lbs is attached. What is the spring constant?

► **Solution.** Recall that the restoring force of the spring balances the gravitational force. Thus $-k(\frac{8}{12}) + 4 = 0$. This gives $k = 6$. The units are lb/ft. ◄

Example 3.8.2. Suppose a body with mass 2 kg stretches a spring 30 centimeters from its natural length. Find the spring constant.

► **Solution.** The force due to gravity is $F_G = 2 \cdot 9.8 = 19.6N$. We now have $-k(\frac{30}{100}) + 19.6 = 0$. This gives $k = \frac{19.6}{.3} = 65.3$. The units are N/m. ◄

Example 3.8.3. Suppose a 6 lb body stretches a spring 2 inches in a frictionless system. Suppose that the body is pulled 3 inches below the spring-body equilibrium and released. Find the motion of the body and provide a graph.

► **Solution.** First let's compute the spring constant. We have $-k\frac{2}{12}+6=0$ so $k=36$ (lb/ft). A frictionless system means that the damping constant is zero. (Of course, this is idealized.) No external force is mentioned so it is zero. Thus the motion is undamped and free. If y measures the displacement from spring-body equilibrium then $y(0)=\frac{3}{12}=\frac{1}{4}$. The body being released implies that the initial velocity is zero. Thus $y'(0)=0$. The mass of the body is 6/32 slugs. The initial value problem thus becomes

$$\frac{6}{32}y''+36y=0, \qquad y(0)=\frac{1}{4}, \quad y'(0)=0.$$

A short calculation gives:

$$y(t)=\frac{1}{4}\cos(8\sqrt{3}t).$$

Figure 3.7 gives the graph. ◄

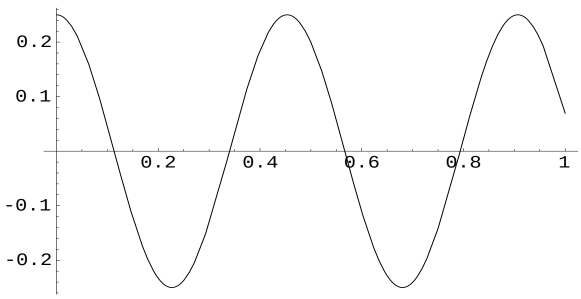


Figure 3.7: Undamped harmonic motion: Graph of $y(t)=.25\cos(8\sqrt{3}t)$

Example 3.8.4. A spring is stretched 49 cm when a 1 kg mass is attached. The body is pulled to 20 cm below its spring-body equilibrium and pushed downward with an initial velocity of 50 cm/sec. There are no external forces. Find the motion of the body when the damping constant is a) 4 N/m. Determine the maximum displacement. b) 12 N/m . In each case provide graphs that represent the motion.

► **Solution. a)** The equation to calculate the spring constant is $1(9.8)-k\frac{49}{100}$ which implies that $k=20$. With $\mu=4$ we have

$$y''+4y'+20y=0 \qquad y(0)=.2, \quad y'(0)=.5.$$

The solution is

$$y=e^{-2t}(\frac{1}{5}\cos 4t+\frac{9}{40}\sin 4t).$$

This represents underdamped harmonic motion whose graph is given in Figure 3.8. The maximum displacement occurs when the derivative is first zero, i.e. $t=0.095$. The corresponding displacement is 0.223 meters. ◄

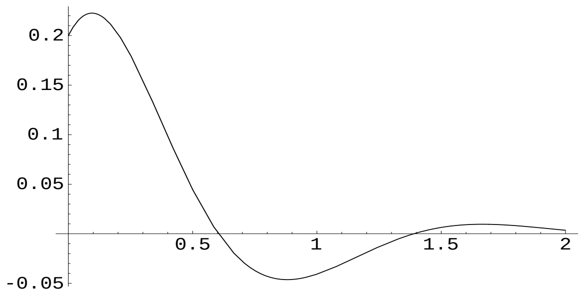


Figure 3.8: Underdamped free motion: Graph of $y(t) = e^{-2t}(\frac{1}{5}\cos 4t + \frac{9}{40}\sin 4t)$.

▶ **Solution. b)** In this case the initial value problem is

$$y'' + 12y' + 20y = 0 \qquad y(0) = .2, \quad y'(0) = .5$$

and the solution is

$$y(t) = \frac{-9}{80}e^{-10t} + \frac{5}{16}e^{-2t}.$$

The graph is given in Figure 3.9. This represents overdamped free motion. The maximal

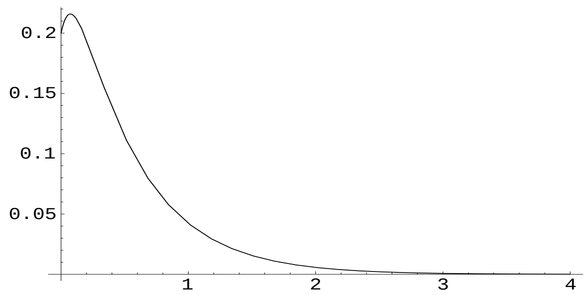


Figure 3.9: Overdamped free motion: Graph of $y(t) = \frac{-9}{80}e^{-10t} + \frac{5}{16}e^{-2t}$.

displacement of 0.216 meters occurs at $t = 0.0735$ seconds. ◀

Exercises

In each of these exercises it is assumed that there is no external force.

An 8-lb weight stretches a spring 1 ft. A 16-lb weight is then attached to the spring, and it comes to rest at the body-spring equilibrium. It is put into motion from equilibrium at a downward velocity of 2 ft/sec.

1. Assume there is no resistance. Determine the motion of the body. What is the maximum displacement?

2. Assume that the damping constant is $k = 2$ lbs/ft. Determine the motion of the body. What is the maximum displacement?

3. Assume that the damping constant is $k = 4$ lbs/ft. Determine the motion of the body. What is the maximum displacement?

4. Assume that the damping constant is $k = 5$ lbs/ft. Determine the motion of the body. What is the maximum displacement?

Chapter 4

DISCONTINUOUS FUNCTIONS AND THE LAPLACE TRANSFORM

For many applications the set of elementary functions, as defined in Chapter 2, or even the set of continuous functions is not sufficiently large to deal with some of the common applications we encounter. Consider two examples. Imagine a mixing problem (see Example 1.1.9 in Section 1.1 and the discussion that follows for a review of mixing problems) where there are two sources of incoming salt solutions with different concentrations. Initially, the first source may be flowing for several minutes. Then the second source is turned on at the same time the first source is turned off. The graph of the input function may well be represented by Figure 4.1. The most immediate observation

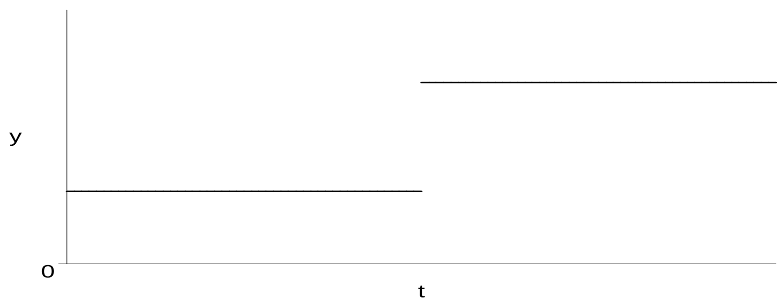


Figure 4.1: A discontinuous input function

is that the input function is discontinuous. Nevertheless, the Laplace transform methods we will develop will easily handle this situation, leading to a formula for the amount of the salt in the tank as a function of time. As a second example, consider a sudden force that is applied to a spring-mass-dashpot system (see Section 3.8 for a discussion of spring-mass-dashpot systems). To model this we imagine that a large force is applied over a small interval. As the interval gets smaller the force gets larger so that the total force always remains a constant. By a limiting process we obtain an instantaneous input called an impulse function. This idea is graphed in Figure 4.2. Such impulse functions

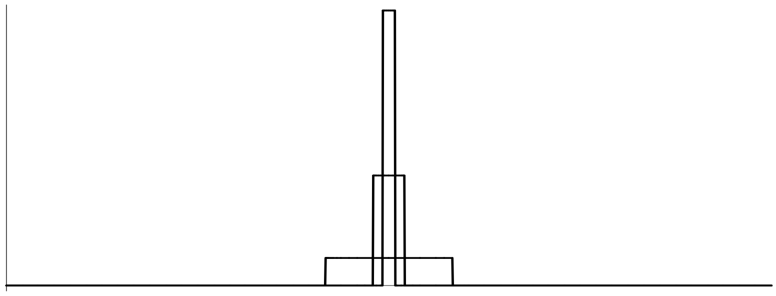


Figure 4.2: An impulse function

have predicable effects on the system. Again the Laplace transform methods we develop here will lead us to the motion of the body without much difficulty.

These two examples illustrate the need to extend the Laplace transform beyond the set of elementary functions that we discussed in Chapter 2. We will do this in two stages. First we will identify a suitably larger class of functions, the Heaviside class, that includes discontinuous functions and then extend the Laplace transform method to this larger class. Second, we will consider the Dirac delta function, which models the impulse function we discussed above. Even though it is called a function the Dirac delta function is actually not a function at all. Nevertheless, its Laplace transform can be defined and the Laplace transform method can be extended to differential equations that involve impulse functions.

4.1 Calculus of Discontinuous Functions

Our focus in the next few sections is a study of first and second order linear constant coefficient differential equations with possibly discontinuous input or forcing function f:

$$y' + ay = f(t)$$
$$y'' + ay' + by = f(t)$$

Allowing f to have some discontinuities introduces some technical difficulties as to what we mean by a solution to such a differential equation. To get an idea of these difficulties and motivate some of the definitions that follow we consider two elementary examples.

First, consider the simple differential equation

$$y' = f(t),$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < \infty. \end{cases}$$

Simply stated what we are seeking is a function y whose derivative is the discontinuous function f. If y is a solution then y must also be a solution when restricted to any subinterval. In particular, let's restrict to the subintervals $(0, 1)$ and $(1, \infty)$, where f is continuous separately. On the interval $(0, 1)$ we obtain $y(t) = c_1$, where c_1 is a constant and on the interval $(1, \infty)$ the solution is $y(t) = t + c_2$, where c_2 is a constant. Piecing these solutions together gives

$$y = \begin{cases} c_1 & \text{if } 0 < t < 1 \\ t + c_2 & \text{if } 1 < t < \infty. \end{cases}$$

Notice that this family has two arbitrary parameters, c_1 and c_2 and unless c_1 and c_2 are chosen just right y will not extend to a continuous function. In applications, like the mixing problem introduced in the introduction, it is reasonable to seek a continuous solution. Thus suppose an initial condition is given, $y(0) = 1$, say, and suppose we wish to find a continuous solution. Since $\lim_{t\to 0^+} y(t) = c_1$, continuity and the initial condition forces $c_1 = 1$ and therefore $y(t) = 1$ on the interval $[0, 1)$. Now since $\lim_{t\to 1^-} y(t) = 1$, continuity forces that we define y(1)=1. Repeating this argument we have $\lim_{t\to 1^+} y(t) = 1 + c_2$ and this forces $c_2 = 0$. Therefore $y(t) = t$ on the interval $(1, \infty)$. Putting these pieces together gives a continuous solution whose graph is given in Figure 4.3. Nevertheless, no matter how we choose the constants c_1 and c_2 the "solution" y is never

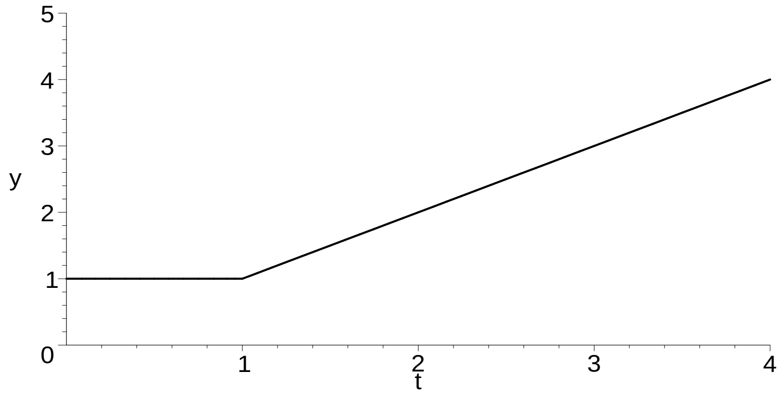


Figure 4.3: A continuous solution to $y' = f(t)$.

differentiable at the point $t = 1$. Therefore the best that we can expect for a solution to $y' = f(t)$ is a continuous function y which is differentiable at all points except $t = 1$.

As a second example consider the differential equation

$$y' = f(t),$$

where

$$f(t) = \begin{cases} \frac{1}{(1-t)^2} & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < \infty. \end{cases}$$

We approach this problem as we did above and obtain that a solution must have the form

$$y(t) = \begin{cases} \frac{1}{1-t} + c_1 & \text{if } 0 < t < 1 \\ t + c_2 & \text{if } 1 < t < \infty, \end{cases}$$

where c_1 and c_2 are arbitrary constants. The graph of this function for $c_1 = 1$ and $c_2 = 1$ is given in Figure 4.4. For us this situation is very undesirable in that no matter how we choose the constants, the solution y will always be discontinuous at $t = 1$. The asymptotic behavior at $t = 1$ for the solution results in the fact that f has a vertical asymptote at $t = 1$.

These examples illustrate the need to be selective in the kinds of discontinuities we allow. In particular, we will require that if f does have a discontinuity it must be a jump discontinuity. The function f in our first example has a jump discontinuity at $t = 1$ while in the second example the discontinuity at $t = 1$ is a result of a vertical asymptote. We also must relax our definition of what we mean by a solution to allow solutions y that have some points where the derivative may not exist. We will be more precise about this later.

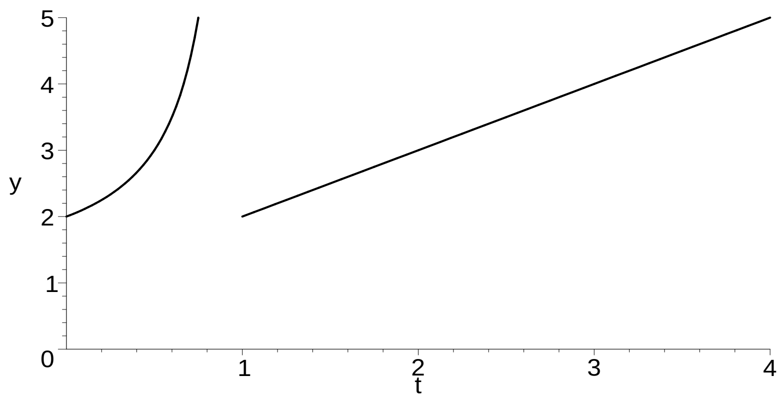


Figure 4.4: Always a discontinuous solution

Jump Discontinuities

We say f has a **jump discontinuity** at a point a if

$$f(a^+) \neq f(a^-)$$

where $f(a^+) = \lim_{t\to a^+} f(t)$ and $f(a^-) = \lim_{t\to a^-} f(t)$. In other words, the left hand limit and the right hand limit at a exist but are not equal. Examples of such functions are typically given **piecewise**, that is, a different formula is used to define f on different subintervals of the domain. For example, consider

$$f(t) = \begin{cases} t^2 & \text{if } 0 \leq t < 1, \\ 1-t & \text{if } 1 \leq t < 2, \\ 1 & \text{if } 2 \leq t \leq 3 \end{cases}$$

whose graph is given in Figure 4.5. We see that f is defined on the interval $[0,3]$ and has a jump discontinuity at $a = 1$ and $a = 2$: $f(1^-) = 1 \neq f(1^+) = 0$ and $f(2^-) = -1 \neq f(2^+) = 1$. On the other hand, the function

$$g(t) = \begin{cases} t & \text{if } 0 \leq t < 1, \\ \frac{1}{t-1} & \text{if } 1 \leq t \leq 2, \end{cases}$$

whose graph is given in Figure 4.6, is defined on the interval $[0,2]$ and has a discontinuity at $a = 1$. However, this is **not** a jump discontinuity because $\lim_{t\to 1^+} g(t)$ does not exist.

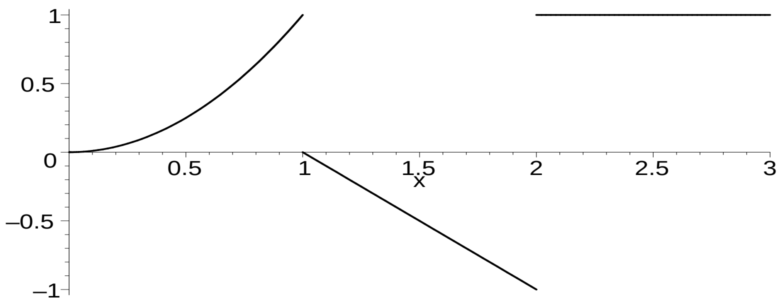


Figure 4.5: A piecewise continuous function

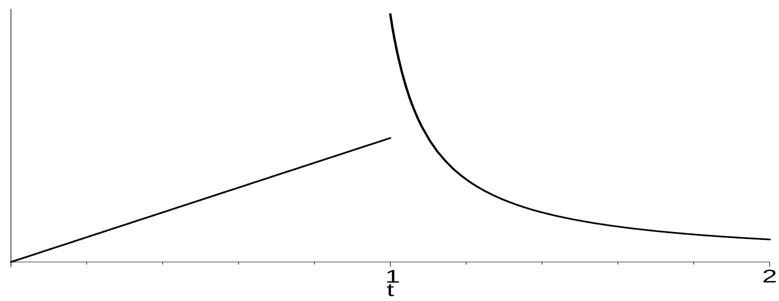


Figure 4.6: A discontinuous function but not a jump discontinuity

For our purposes we will say that a function f is **piecewise continuous on an interval** $[\alpha, \beta]$ if f is continuous except for possibly finitely many jump discontinuities. If an interval is not specified it will be understood that f is defined on $[0, \infty)$ and f is continuous on all subintervals of the form $[0, N]$ except for possibly finitely many jump discontinuities. For convenience it will not be required that f be defined at the jump discontinuities. Suppose $a_1, \ldots, a_n$ are the locations of the jump discontinuities in the interval $[0, N]$ and assume $a_i < a_{i+1}$, for each i. On the interval (a_i, a_{i+1}) we can extend f to a continuous function on the closed interval $[a_i, a_{i+1}]$. Since a continuous function on a closed interval is bounded and there are only finitely many jump discontinuities we have the following proposition.

Proposition 4.1.1. *If f is a piecewise continuous function on $[0, N]$ then f is bounded.*

Integration of Piecewise Continuous Functions

Suppose f is a piecewise continuous function on the interval $[0, N]$ and the jump discontinuities are located at $a_1, \ldots, a_k$. We may assume $a_i < a_{i+1}$ and we will let $a_0 = 0$ and $a_{k+1} = N$. In this case the definite integral of f on $[0, N]$ exists and

$$\int_0^N f(t)\,dt = \int_{a_0}^{a_1} f(t)\,dt + \int_{a_1}^{a_2} f(t)\,dt + \cdots + \int_{a_k}^{a_{k+1}} f(t)\,dt.$$

On each interval of the form (a_i, a_{i+1}) f is continuous and therefore an antiderivative F_i exists. Since f is bounded so is F_i and thus may be extended to the closed interval $[a_i, a_{i+1}]$. When necessary we will denote the extended values of F_i at a_i by $F_i(a_i^+)$ and at a_{i+1} by $F_i(a_{i+1}^-)$. We then have

$$\int_{a_i}^{a_{i+1}} f(t)\,dt = F_i(a_i^+) - F_i(a_{i+1}^-).$$

Example 4.1.2. Find $\int_{-1}^5 f(t)\,dt$, if

$$f(t) = \begin{cases} t & \text{if } -1 \le t < 1 \\ 2 & \text{if } t = 1 \\ \frac{1}{t} & \text{if } 1 \le t < 3 \\ 2 & \text{if } 3 \le t < 5 \end{cases}$$

▶ **Solution.** The function f is piecewise continuous and

$$\begin{aligned} \int_{-1}^5 f(t)\,dt &= \int_{-1}^1 t\,dt + \int_1^3 \frac{1}{t}\,dt + \int_3^5 2\,dt \\ &= 0 + \ln 3 + 4 = 4 + \ln 3. \end{aligned}$$

We note that the value of f at $t = 1$ played no role in the computation of the integral. ◀

Example 4.1.3. Find $\int_0^t f(u)\,du$ for the piecewise function f given by

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 1, \\ 1 - t & \text{if } 1 \le t < 2, \\ 1 & \text{if } 2 \le t < \infty. \end{cases}$$

▶ **Solution.** The function f is given piecewise on the intervals $[0, 1)$, $[1, 2)$ and $[2, \infty)$. We will therefore consider three cases. If $t \in [0, 1)$ then

$$\int_0^t f(u)\,du = \int_0^t u^2\,du = \frac{t^3}{3}.$$

It $t \in [1, 2)$ then

$$\begin{aligned}\int_0^t f(u)\,du &= \int_0^1 f(u)\,du + \int_1^t f(u)\,du \\ &= \frac{1}{3} + \int_1^t (1-u)\,du \\ &= \frac{1}{3} + \left(t - \frac{t^2}{2} - \frac{1}{2}\right) = -\frac{t^2}{2} + t - \frac{1}{6}.\end{aligned}$$

Finally, if $t \geq 2$ then

$$\begin{aligned}\int_0^t f(u)\,du &= \int_0^2 f(u)\,du + \int_2^t f(u)\,du \\ &= -\frac{1}{6} + \int_2^t 1\,du = t - \frac{13}{6}\end{aligned}$$

Piecing these functions together gives

$$\int_0^t f(u)\,du = \begin{cases} \frac{t^3}{3} & \text{if } 0 \leq t < 1 \\ -\frac{t^2}{2} + t - \frac{1}{6} & \text{if } 1 \leq t < 2 \\ t - \frac{13}{6} & \text{if } 2 \leq t < \infty. \end{cases}$$

It is continuous as can be observed in the graph given in Figure 4.7 ◄

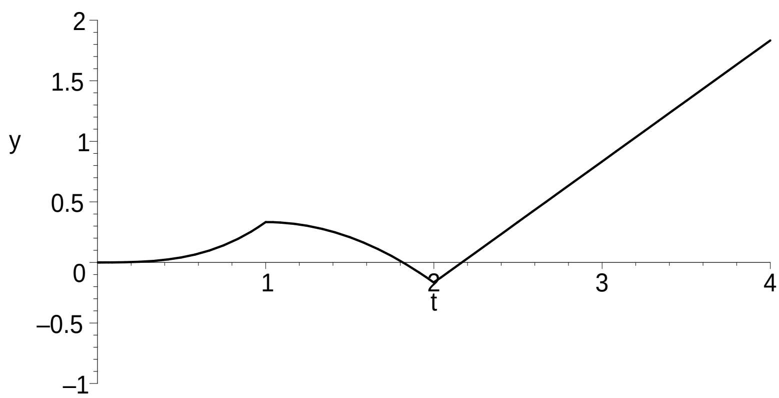


Figure 4.7: The graph of the integral of the discontinuous function in Example 4.1.3

The discussion above leads to the following proposition.

Proposition 4.1.4. *If f is a piecewise continuous function on an interval $[\alpha, \beta]$ and $a, t \in [\alpha, \beta]$ then the integral $\int_a^t f(u)\,du$ exists and is a continuous function in the variable t.*

Proof. The integral exists as discussed above. Let $F(t) = \int_a^t f(u)\,du$. Since f is piecewise continuous on $[\alpha, \beta]$ it is bounded by Proposition 4.1.1. We may then suppose $|f(t)| \leq B$, for some $B > 0$. Let $\epsilon > 0$. Then

$$|F(t+\epsilon) - F(t)| \leq \int_t^{t+\epsilon} |f(u)|\,du \leq \int_t^{t+\epsilon} B\,du = B\epsilon.$$

Therefore $\lim_{\epsilon \to 0} F(t+\epsilon) = F(t)$ and hence $F(t^+) = F(t)$. In a similar way $F(t^-) = F(t)$. This establishes the continuity of F. □

Differentiation of Piecewise Continuous Functions

In the applications, we will consider (continuous) functions that are differentiable on intervals $[0, N)$ except at finitely many points. In this case we will use the symbol f' to denote the derivative of f though it may not be defined at some points. For example, consider

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ t-1 & \text{if } 1 \leq t < \infty. \end{cases}$$

This function is continuous on $[0, \infty)$ and differentiable at all points except $t = 1$. A simple calculation gives

$$f'(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ 1 & \text{if } 1 < t < \infty. \end{cases}$$

Notice that f' is not defined at $t = 1$.

Differential Equations and Piecewise Continuous Functions

We are now in a position to consider some examples of constant coefficient linear differential equations with piecewise continuous forcing functions. Let $f(t)$ be a piecewise continuous function. We say that a function y is a **solution** to $y' + ay = f(t)$ if y is continuous and satisfies the differential equation except at the location of the discontinuities of the input function.

Example 4.1.5. Find a continuous solution to

$$y' + 2y = f(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ t-1 & \text{if } 1 \le t < 3 \\ 0 & \text{if } 3 \le t < \infty, \end{cases} \qquad y(0) = 1. \tag{1}$$

► **Solution.** We will apply the method that we discussed at the beginning of this section. That is, we will consider the differential equation on the subintervals where f is continuous and then piece the solution together. In each case the techniques discussed in Section 1.3 apply. The first subinterval is $(0, 1)$, where $f(t) = 1$. Multiplying both sides of $y' + 2y = 1$ by the integrating factor, e^{2t}, leads to $(e^{2t}y)' = e^{2t}$. Integrating and solving for y gives $y = \frac{1}{2} + ce^{-2t}$. The initial condition $y(0) = 1$ implies that $c = \frac{1}{2}$ and therefore

$$y = \frac{1}{2} + \frac{1}{2}e^{-2t}, \quad 0 \le t < 1.$$

We let $y(1) = y(1^-) = \frac{1}{2}+\frac{1}{2}e^{-2}$. This value becomes the initial condition for $y'+2y = t-1$ on the interval $[1, 3)$. Following a similar procedure leads to

$$y = \frac{t-1}{2} - \frac{1}{4} + \frac{3}{4}e^{-2(t-1)} + \frac{1}{2}e^{-2t}, \quad 1 \le t < 3.$$

We define $y(3) = y(3^-) = \frac{3}{4} + \frac{3}{4}e^{-4} + \frac{1}{2}e^{-6}$. This value becomes the initial condition for $y' + 2y = 0$ on the interval $[3, \infty)$. Its solution there is the function

$$y = \frac{3}{4}e^{-2(t-3)} + \frac{3}{4}e^{-2(t-1)} + \frac{1}{2}e^{-2t}.$$

Putting these pieces together gives the solution

$$y(t) = \begin{cases} \frac{1}{2} + \frac{1}{2}e^{-2t} & \text{if } 0 \le t < 1 \\ \frac{t-1}{2} - \frac{1}{4} + \frac{3}{4}e^{-2(t-1)} + \frac{1}{2}e^{-2t} & \text{if } 1 \le t < 3 \\ \frac{3}{4}e^{-2(t-3)} + \frac{3}{4}e^{-2(t-1)} + \frac{1}{2}e^{-2t} & \text{if } 3 \le t < \infty. \end{cases}$$

By making the initial value on each subinterval the left hand limit of the solution on the previous interval we guarantee continuity. The graph of this solution is shown in Figure 4.8. The discontinuity of the derivative of y at $t = 1$ and $t = 3$ is evident by the kinks at those points. ◄

The method we used here insures that the solution we obtain is continuous and the initial condition at $t = 0$ determines the subsequent initial conditions at the points of discontinuity of f. We also note that the initial condition at $t = 0$, the left hand endpoint of the domain, was chosen only for convenience; we could have taken the initial value at any point $t_0 \ge 0$ and pieced together a continuous function on both sides of t_0. That this can be done in general is stated in the following theorem.

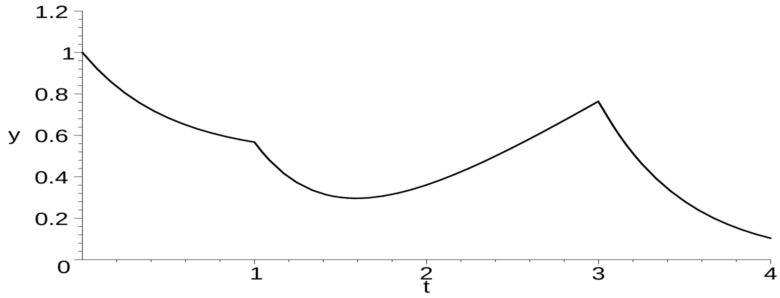


Figure 4.8: The graph of the solution to Example 4.1.5

Theorem 4.1.6. *Suppose f is a piecewise continuous function on an interval $[\alpha, \beta]$ and $t_0 \in [\alpha, \beta]$. There is a unique continuous function y which satisfies*

$$y' + ay = f(t), \qquad y(t_0) = y_0.$$

Recall that this means that y' will not exist at the points of discontinuity of f.

Proof. We follow the method illustrated in the example above to construct a continuous solution. To prove uniqueness suppose y_1 and y_2 are two continuous solutions. If $y = y_1 - y_2$ then $y(t_0) = 0$ and y is a continuous solution to

$$y' + ay = 0.$$

On the interval containing t_0 on which f is continuous, $y = 0$ by the uniqueness and existence theorem. The initial value at the endpoint of adjacent intervals is thus 0. Continuing in this way we see that y is identically 0 on $[\alpha, \beta]$ and hence $y_1 = y_2$. □

We now consider a second order constant coefficient differential equation with a piecewise continuous forcing function. Our method is similar to the one above, however, we demand more out of our solution. If $f(t)$ be a piecewise continuous function then we say a function y is a **solution** to $y'' + ay' + by = f(t)$ if y is continuous, has a continuous derivative, and satisfies the differential equation except at the discontinuities of the forcing function f.

Example 4.1.7. Find a solution y to

$$y'' + y = f(t) = \begin{cases} t & \text{if } 0 \le t < 2\pi \\ 2 & \text{if } 2\pi \le t \le 4\pi, \end{cases} \qquad y(0) = 0 \; y'(0) = 1.$$

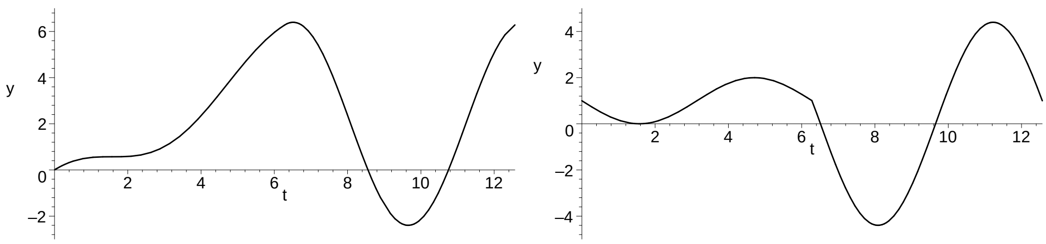
▶ **Solution.** We begin by considering the differential equation $y'' + y = t$ on the interval $[0, 2\pi)$. The homogenous solution is $y_h(t) = a\cos t + b\sin t$ and the method of undetermined coefficients or variation of parameters leads to a particular solution $y_p(t) = t - 1$. The general solution is $y = t - 1 + a\cos t + b\sin t$ and incorporating the initial conditions leads to $y = t - 1 + \cos t$ on the interval $[0, 2\pi)$. We calculate that $y' = 1 - \sin t$. In order to piece together a solution that is continuous at $t = 2\pi$ we must have $y(2\pi) = y(2\pi^-) = 2\pi$. In order for the derivative y' to be continuous at $t = 2\pi$ we must have $y'(2\pi) = y'(2\pi^-) = 1$. We use these values for the initial conditions on the interval $[2\pi, 4\pi]$. The general solution to $y'' + y = 2$ is $y = 2 + a\cos t + b\sin t$. The initial conditions imply $a = 2\pi - 2$ and $b = 1$ and thus $y = 2 + (2\pi - 2)\cos t + \sin t$. Piecing these two solutions together gives

$$y(t) = \begin{cases} t - 1 + \cos t & \text{if } 0 \le t < 2\pi \\ 2 + (2\pi - 2)\cos t + \sin t & \text{if } 2\pi \le t \le 4\pi. \end{cases}$$

Its derivative is

$$y'(t) = \begin{cases} 1 - \sin t & \text{if } 0 \le t < 2\pi \\ -(2\pi - 2)\sin t + \cos t & \text{if } 2\pi \le t \le 4\pi. \end{cases}$$

Figure 4.9 gives (a) the graph of the solution and (b) the graph of its derivative. The



(a) (b)

Figure 4.9: The solution (a) and its derivative (b) to Example 4.1.7

solution is differentiable on the interval $[0, 4\pi]$ and the derivative is continuous on $[0, 4\pi)$. However, the kink in the derivative at $t = 2\pi$ indicates that the second derivative is not continuous. ◀

In direct analogy to the first order case we considered above we are lead to the following theorem. The proof is omitted.

Theorem 4.1.8. *Suppose f is a piecewise continuous function on an interval $[\alpha, \beta]$ and $t_0 \in [\alpha, \beta]$. There is a unique continuous function y which satisfies*

$$y'' + ay' + by = f(t), \qquad y(t_0) = y_0,\ y'(t_0) = y_1.$$

Furthermore, y is differentiable and y' is continuous.

Piecing together solutions in the way that we described above is at best tedious. As we proceed we will extend the Laplace transform to a class of functions that includes piecewise continuous function. The Laplace transform method extends as well and will provide an alternate method for solving differential equations like the ones above. It is one of the hallmarks of the Laplace transform.

Exercises

Match the following functions that are given piecewise with their graphs and determine where jump discontinuities occur.

1. $f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 4 \\ -1 & \text{if } 4 \leq t < 5 \\ 0 & \text{if } 5 \leq t < \infty. \end{cases}$

2. $f(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 2-t & \text{if } 1 \leq t < 2 \\ 1 & \text{if } 2 \leq t < \infty. \end{cases}$

3. $f(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 2-t & \text{if } 1 \leq t < \infty. \end{cases}$

4. $f(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ t-1 & \text{if } 1 \leq t < 2 \\ t-2 & \text{if } 2 \leq t < 3 \\ \vdots & . \end{cases}$

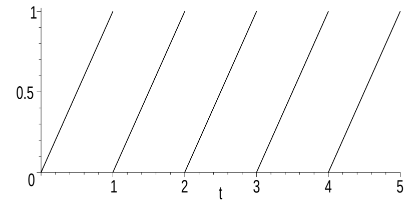
5. $f(t) = \begin{cases} 1 & \text{if } 2n \leq t < 2n+1 \\ 0 & \text{if } 2n+1 \leq t < 2n+2. \end{cases}$

6. $f(t) = \begin{cases} t^2 & \text{if } 0 \leq t < 2 \\ 4 & \text{if } 2 \leq t < 3 \\ 7-t & \text{if } 3 \leq t < \infty. \end{cases}$

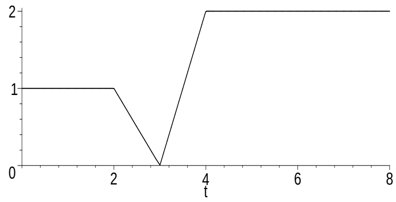
7. $f(t) = \begin{cases} 1-t & \text{if } 0 \le t < 2 \\ 3-t & \text{if } 2 \le t < 4 \\ 5-t & \text{if } 4 \le t < 6 \\ \vdots & . \end{cases}$

8. $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ 3-t & \text{if } 2 \le t < 3 \\ 2(t-3) & \text{if } 3 \le t < 4 \\ 2 & \text{if } 4 \le t < \infty. \end{cases}$

Graphs for problems 1 through 8



(a)



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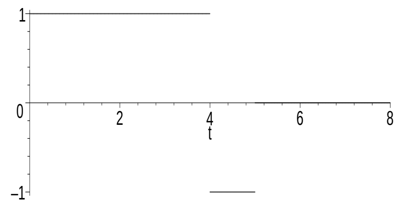

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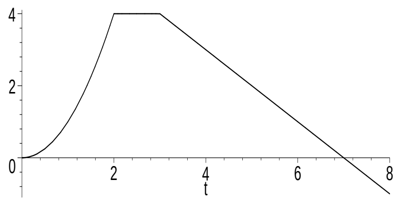

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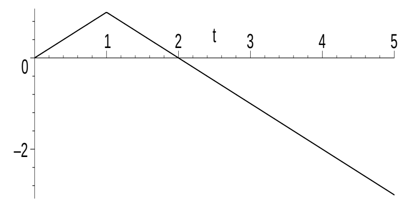

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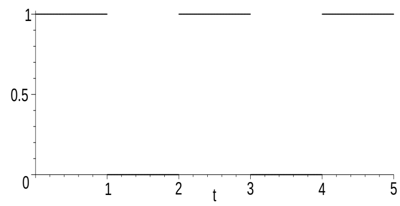
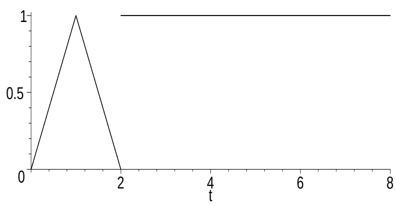
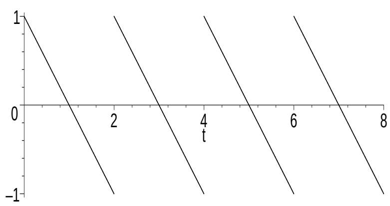


(f)



(g)



(h)

In problems 9 through 12 calculate the indicated integral.

9. $\int_0^5 f(t)\,dt$, where $f(t) = \begin{cases} t^2 - 4 & \text{if } 0 \le t < 2 \\ 0 & \text{if } 2 \le t < 3 \\ -t+3 & \text{if } 3 \le t < 5. \end{cases}$

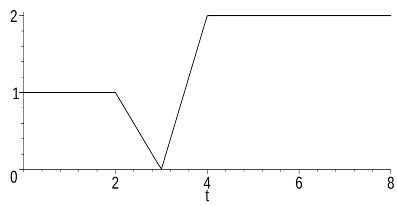
10. $\int_0^2 f(u)\,du$, where $f(u) = \begin{cases} 2-u & \text{if } 0 \le u < 1 \\ u^3 & \text{if } 1 \le u < 2. \end{cases}$

11. $\int_0^{2\pi} |\sin(x)|\ dx$.

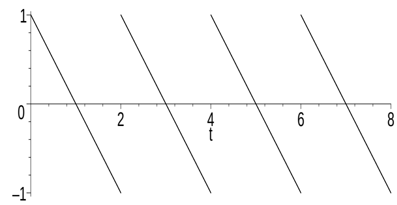
12. $\int_0^3 f(w)\,dw$ where $f(w) = \begin{cases} w & \text{if } 0 \le w < 1 \\ \frac{1}{w} & \text{if } 1 \le w < 2 \\ \frac{1}{2} & \text{if } 2 \le w < \infty. \end{cases}$

In problems 13 through 16 find the indicated integral. (See problems 1 through 9 for the appropriate formula.)

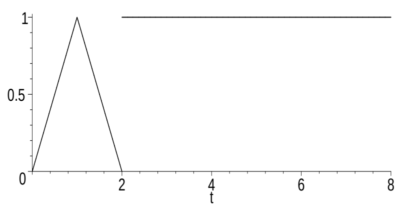
13. $\int_2^5 f(t)\,dt$, where the graph of f is:



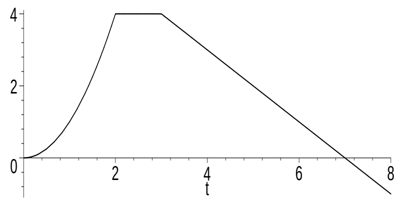
14. $\int_0^8 f(t)\,dt$, where the graph of f is:



15. $\int_0^6 f(u)\,du$, where the graph of f is:



16. $\int_0^7 f(t)\,dt$, where the graph of f is:



17. Of the following four piecewise defined functions determine which ones (A) satisfy the differential equation

$$y' + 4y = f(t) = \begin{cases} 4 & \text{if } 0 \le t < 2 \\ 8t & \text{if } 2 \le t < \infty, \end{cases}$$

except at the point of discontinuity of f, (B) are continuous, and (C) are continuous solutions to the differential equation with initial condition $y(0) = 2$. Do not solve the differential equation.

(a) $y(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ 2t - \frac{1}{2} - \frac{5}{2}e^{-4(t-2)} & \text{if } 2 \le t < \infty \end{cases}$

(b) $y(t) = \begin{cases} 1 + e^{-4t} & \text{if } 0 \le t < 2 \\ 2t - \frac{1}{2} - \frac{5}{2}e^{-4(t-2)} + e^{-4t} & \text{if } 2 \le t < \infty \end{cases}$

(c) $y(t) = \begin{cases} 1 + e^{-4t} & \text{if } 0 \le t < 2 \\ 2t - \frac{1}{2} - \frac{5e^{-4(t-2)}}{2} & \text{if } 2 \le t < \infty \end{cases}$

(d) $y(t) = \begin{cases} 2e^{-4t} & \text{if } 0 \le t < 2 \\ 2t - \frac{1}{2} - \frac{5}{2}e^{-4(t-2)} + e^{-4t} & \text{if } 2 \le t < \infty \end{cases}$

18. Of the following four piecewise defined functions determine which ones (A) satisfy the differential equation

$$y'' - 3y' = 2y = f(t) = \begin{cases} e^t & \text{if } 0 \le t < 1 \\ e^{2t} & \text{if } 1 \le t < \infty, \end{cases}$$

except at the point of discontinuity of f, (B) are continuous, and (C) have continuous derivatives, and (D) are continuous solutions to the differential equation with initial conditions $y(0) = 0$ and $y'(0) = 0$ and have continuous derivatives. Do not solve the differential equation.

(a) $y(t) = \begin{cases} -te^t - e^t + e^{2t} & \text{if } 0 \le t < 1 \\ te^{2t} - 2e^t & \text{if } 1 \le t < \infty \end{cases}$

(b) $y(t) = \begin{cases} -te^t - e^t + e^{2t} & \text{if } 0 \le t < 1 \\ te^{2t} - 3e^t - \frac{1}{2}e^{2t} & \text{if } 1 \le t < \infty \end{cases}$

(c) $y(t) = \begin{cases} -te^t - e^t + e^{2t} & \text{if } 0 \le t < 1 \\ te^{2t} + e^{t+1} - e^t - e^{2t} - e^{2t-1} & \text{if } 1 \le t < \infty \end{cases}$

(d) $y(t) = \begin{cases} -te^t + e^t - e^{2t} & \text{if } 0 \le t < 1 \\ te^{2t} + e^{t+1} + e^t - e^{2t-1} - 3e^{2t} & \text{if } 1 \le t < \infty \end{cases}$

Solve the following differential equations.

19. $y' + 3y = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < \infty, \end{cases} \qquad y(0) = 0.$

20. $y' - y = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ t-1 & \text{if } 1 \le t < 2 \\ 3-t & \text{if } 2 \le t < 3 \\ 0 & \text{if } 3 \le t < \infty, \end{cases} \qquad y(0) = 0.$

21. $y' + y = \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } \pi \le t < \infty \end{cases} \qquad y(\pi) = -1.$

22. $y'' - y = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < \infty, \end{cases} \qquad y(0) = 0,\ y'(0) = 1.$

23. $y'' - 4y' + 4y = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ 4 & \text{if } 2 \le t < \infty \end{cases} \qquad y(0) = 1,\ y'(0) = 0$

24. Suppose f is a piecewise continuous function on an interval $[\alpha, \beta]$. Let $a \in [\alpha, \beta]$ and define $y(t) = y_0 + \int_a^t f(u)\, du$. Show that y is a continuous solution to

$$y' = f(t) \quad y(a) = y_0.$$

25. Suppose f is a piecewise continuous function on an interval $[\alpha, \beta]$. Let $a \in [\alpha, \beta]$ and define $y(t) = y_0 + e^{-at} \int_a^t e^{au} f(u)\, du$. Show that y is a continuous solution to

$$y' + ay = f(t) \quad y(a) = y_0.$$

26. Let $f(t) = \begin{cases} \sin(1/t) & \text{if } t \ne 0 \\ 0 & \text{if } t = 0. \end{cases}$

   (a) Show that f is bounded.

   (b) Show that f is not continuous at $t = 0$.

   (c) Show that f is not piecewise continuous.

4.2 The Heaviside class $\mathcal{H}$

In this section we will extend the definition of the Laplace transform beyond the set of elementary functions $\mathcal{E}$ to include piecewise continuous functions. The Laplace transform method will extend as well and provide a rather simple means of dealing with the differential equations we saw in Section 4.1.

Since the Laplace transform

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

is defined by means of an improper integral, we must be careful about the issue of convergence. Recall that this definition means we compute $\int_0^N e^{-st} f(t)\,dt$ and then take the limit as N goes to infinity. To insure convergence we must take into consideration the kinds of functions we feed into it. There are two main reasons why such improper integrals may fail to exist. First, if the distribution of the discontinuities of f is 'too bad' then even the finite integral $\int_0^N e^{-st} f(t)\,dt$ may fail to exist. Second, if the finite integral exists the limit as N goes to ∞ may not. This has to do with how fast f grows. These two issues will be handled separately by (1) identifying the particular type of discontinuities allowed and (2) restricting the type of growth that f is allowed. What will result is a class of functions large enough to handle most of the applications one is likely to encounter.

The first issue is handled for us by restricting to the piecewise continuous functions defined in section 4.1. If f is a piecewise continuous function (on $[0,\infty)$) then so is $t \mapsto e^{-st} f(t)$. By Proposition 4.1.4 the integral

$$\int_0^N e^{-st} f(t)\,dt$$

exists and is a continuous function in the variable N. Now to insure convergence as N goes to ∞ we must place a further requirement on f.

Functions of Exponential Type

We now want to put further restrictions on f to assure that $\lim_{N\to\infty} \int_0^N e^{-st} f(t)\,dt$ exists. As we indicated this can be achieved by making sure that f doesn't grow too fast.

A function $y = f(t)$ is said to be of **exponential type** if

$$|f(t)| \leq Ke^{at}$$

for all $t \geq M$, where M, K, and a are positive real constants. The idea here is that functions of exponential type should not grow faster than a multiple of an exponential function Ke^{at}. Visually, we require the graph of $|f|$ to lie below such an exponential function from some point on, $t \geq M$, as illustrated in Figure 4.10.

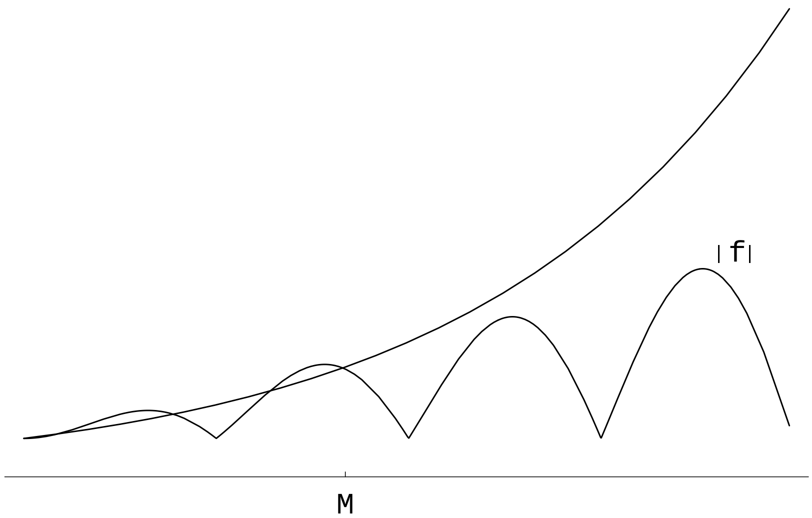


Figure 4.10: The exponential function Ke^{at} eventually overtakes $|f|$ for $t \geq M$.

We note here that in the case where f is also piecewise continuous then f is bounded on $[0, M]$ and one can find a constant K' such that

$$|f(t)| \leq K'e^{at}$$

for all $t > 0$.

The **Heaviside class** is the set $\mathcal{H}$ of all piecewise continuous functions of exponential type. One can show it is closed under addition and scalar multiplication. (see Exercise ??) It is to this class of functions that we extend the Laplace transform. The set of elementary functions $\mathcal{E}$ that we introduced in Chapter 2 are all examples of functions in the Heaviside class. Recall that f is an elementary function if f is a sum of functions of the form $ct^n e^{at} \sin(bt)$ and $ct^n e^{at} \cos(bt)$, where a, b, c are constants and n is a nonnegative integer. Such functions are continuous. Since sin and cos are bounded by 1 and $t^n \leq e^{nt}$ it follows that $|ct^n e^{at} \sin bt| \leq ce^{(a+n)t}$ and likewise $|ct^n e^{at} \cos bt| \leq ce^{(a+n)t}$. Thus elementary functions are of exponential type, i.e., $\mathcal{E} \subset \mathcal{H}$. Although the Heaviside class is much bigger than the set of elementary functions there are many important functions which are not in $\mathcal{H}$. An example is $f(t) = e^{t^2}$. For if b is any positive constant, then

$$\frac{e^{t^2}}{e^{bt}} = e^{t^2-bt} = e^{(t-\frac{b}{2})^2-\frac{b^2}{4}}$$

and therefore,

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{bt}} = \infty.$$

This implies that $f(t) = e^{t^2}$ grows faster than any exponential function and thus is not of exponential type.

Existence of the Laplace transform

Recall that for an elementary function the Laplace transform exists and has the further property that $\lim_{s\to\infty} F(s) = 0$. These two properties extend to all functions in the Heaviside class.

Theorem 4.2.1. *For $f \in \mathcal{H}$ the Laplace transform exists and*

$$\lim_{s\to\infty} F(s) = 0.$$

Proof. The finite integral $\int_0^N e^{-st} f(t)\, dt$ exists because f is piecewise continuous on $[0, N]$. Since f is also of exponential type there are constants K and a such that $|f(t)| \leq Ke^{at}$ for all $t \geq 0$. Thus, for all $s > a$,

$$\begin{aligned} \int_0^\infty |e^{-st} f(t)|\, dt &\leq \int_0^\infty |e^{-st} K e^{at}|\, dt \\ &= K \int_0^\infty e^{-(s-a)t}\, dt \\ &= \frac{K}{s-a}. \end{aligned}$$

This shows that the integral converges absolutely and hence the Laplace transform exists for $s > a$. Since $|\mathcal{L}\{f\}(s)| \leq \frac{K}{s-a}$ and $\lim_{s\to\infty} \frac{K}{s-a} = 0$ it follows that

$$\lim_{s\to\infty} \mathcal{L}\{f\}(s) = 0.$$

□

As might be expected computations using the definition to compute Laplace transforms of even simple functions can be tedious. To illustrate the point consider the following example.

Example 4.2.2. Use the definition to compute the Laplace transform of

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 1 \\ 2 & \text{if } 1 \le t < \infty. \end{cases}$$

► **Solution.** Clearly f is piecewise continuous and bounded, hence it is in the Heaviside class. We can thus proceed with the definition confident, by Theorem 4.2.1, that the improper integral will converge. We have

$$\begin{aligned} \mathcal{L}\{f\}(s) &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^1 e^{-st}\, t^2\,dt + \int_1^\infty e^{-st}\, 2\,dt \end{aligned}$$

For the first integral we need integration by parts twice:

$$\begin{aligned} \int_0^1 e^{-st}\, t^2\,dt &= \frac{t^2 e^{-st}}{-s}\Big|_0^1 + \frac{2}{s}\int_0^1 e^{-st}\, t\,dt \\ &= \frac{e^{-s}}{-s} + \frac{2}{s}\left(\frac{te^{-st}}{-s}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt\right) \\ &= -\frac{e^{-s}}{s} + \frac{2}{s}\left(-\frac{e^{-s}}{s} - \frac{1}{s^2}e^{-st}\Big|_0^1\right) \\ &= -\frac{e^{-s}}{s} - \frac{2e^{-s}}{s^2} + \frac{2}{s^3} - \frac{2e^{-s}}{s^3}. \end{aligned}$$

The second integral is much simpler and we get

$$\int_1^\infty e^{-st}\, 2\,dt = \frac{2e^{-s}}{s}$$

Now putting things together and simplifying gives

$$\mathcal{L}\{f\}(s) = \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}\right).$$

◄

Do not despair. The Heaviside function that we introduce next will lead to a Laplace transform principle that will make unnecessary calculations like the one above.

The Heaviside Function

In order to effectively manage piecewise continuous functions in $\mathcal{H}$ it is useful to introduce an important auxiliary function called the **unit step function** or **Heaviside function**:

$$h_c(t) = \begin{cases} 0 & \text{if } 0 \le t < c, \\ 1 & \text{if } c \le t. \end{cases}$$

The graph of this function is given in Figure 4.11.

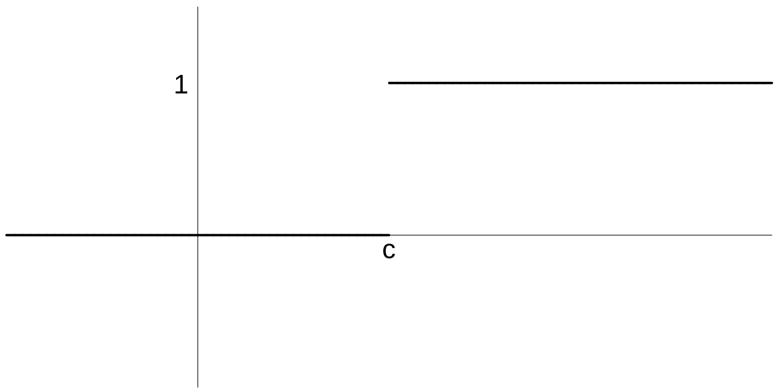


Figure 4.11: The Heaviside Function $h_c(t)$

Clearly, it is piecewise continuous, and since it is bounded it is of exponential type. Thus $h_c \in \mathcal{H}$. Frequently we will write $h(t) = h_0(t)$. Observe also that $h_c(t) = h(t - c)$. More complicated functions can be built from the Heaviside function. First consider the model for an **on-off** switch, $\chi_{[a,b)}$, which is 1 (the on state) on the interval $[a, b)$ and 0 (the off state) elsewhere. Its graph is given in Figure 4.12. Observe that $\chi_{[a,b)} = h_a - h_b$ and $\chi_{[a,\infty)} = h_a$. Now using on-off switches we can easily describe functions defined piecewise.

Example 4.2.3. Write the piecewise defined function

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 1, \\ 2 & \text{if } 1 \le t < \infty. \end{cases}$$

in terms of on-off switches and in terms of Heaviside functions.

► **Solution.** In this piecewise function t^2 is in the on state only in the interval $[0, 1)$ and 2 is in the on state only in the interval $[1, \infty)$.Thus

$$f(t) = t^2 \chi_{[0,1)} + 2\chi_{[1,\infty)}.$$

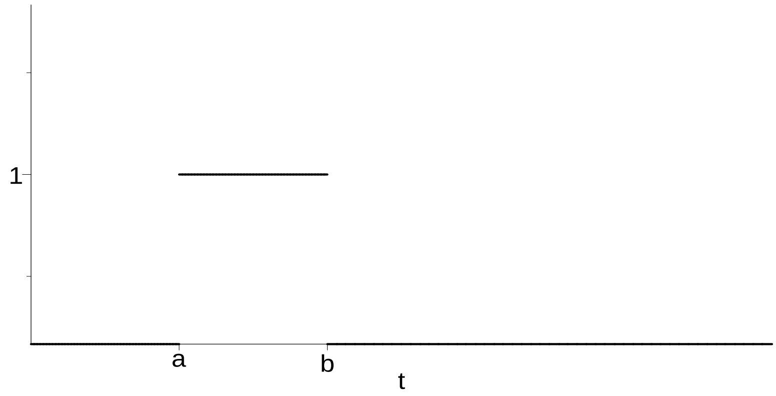


Figure 4.12: The On/Off Switch $\chi_{a,b}(t)$

Now rewriting the on-off switches in terms of the Heaviside functions we obtain:

$$\begin{aligned} f(t) &= t^2(h_0 - h_1) + 2h_1 \\ &= t^2 h_0 + (2 - t^2)h_1 \\ &= t^2 + (2 - t^2)h(t-1). \end{aligned}$$

◀

The Laplace Transform on the Heaviside class

The importance of writing piecewise continuous functions in terms of Heaviside functions is seen by the ease of computing its Laplace transform. For simplicity when $f \in \mathcal{H}$ we will extend f by defining $f(t) = 0$ when $t < 0$. This extension does not effect the Laplace transform for the Laplace transform only involves values $f(t)$ for $t > 0$.

Theorem 4.2.4 (The Second Translation Principle). *Suppose $f \in \mathcal{H}$ is a function with Laplace transform F. Then*

$$\mathcal{L}\{f(t-c)h(t-c)\} = e^{-sc}F(s).$$

In terms of the inverse Laplace transform this is equivalent to

$$\mathcal{L}^{-1}\left\{e^{-sc}F(s)\right\} = f(t-c)h(t-c).$$

Proof. The calculation is straightforward and involves a simple change of variables:

$$\begin{aligned}
\mathcal{L}\{f(t-c)h(t-c)\}(s) &= \int_0^\infty e^{-st} f(t-c)h(t-c)\,dt \\
&= \int_c^\infty e^{-st} f(t-c)\,dt \\
&= \int_0^\infty e^{-s(t+c)} f(t)\,dt \qquad (t \mapsto t+c) \\
&= e^{-sc} \int_0^\infty e^{-st} f(t)\,dt \\
&= e^{-sc} F(s)
\end{aligned}$$

□

Frequently, we encounter expressions in the form $g(t)h(t-c)$. If $f(t)$ is replaced by $g(t+c)$ in Theorem 4.2.4 then we obtain

Corollary 4.2.5.

$$\mathcal{L}\{g(t)h(t-c)\} = e^{-sc}\mathcal{L}\{g(t+c)\}.$$

A simple example of this is when $g = 1$. Then $\mathcal{L}\{h_c\} = e^{-sc}\mathcal{L}\{1\} = \frac{e^{-sc}}{s}$. When $c = 0$ then $\mathcal{L}\{h_0)\} = \frac{1}{s}$ which is the same as the Laplace transform of the constant function 1. This is consistent since $h_0 = h = 1$ for $t \geq 0$.

Example 4.2.6. Find the Laplace transform of $f(t) = \begin{cases} t^2 & \text{if } 0 \leq t < 1 \\ 2 & \text{if } 1 \leq t < \infty \end{cases}$ given in Example 4.2.

► **Solution.** In Example 4.2.3 we found $f(t) = t^2 + (2 - t^2)h(t-1)$. By the Corollary we get

$$\begin{aligned}
\mathcal{L}\{f\} &= \frac{2}{s^3} + e^{-s}\mathcal{L}\left\{2 - (t+1)^2\right\} \\
&= \frac{2}{s^3} + e^{-s}\mathcal{L}\left\{-t^2 - 2t + 1\right\} \\
&= \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}\right)
\end{aligned}$$

◄

Example 4.2.7. Find the Laplace transform of

$$f(t) = \begin{cases} \cos t & \text{if } 0 \le t < \pi \\ 1 & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 2\pi \le t < \infty. \end{cases}$$

▶ **Solution.** First writing f in terms of on-off switches gives

$$f = \cos t \ \chi_{[0,\pi)} + 1 \ \chi_{[\pi,2\pi)} + 0 \ \chi_{[2\pi,\infty)}.$$

Now rewrite this expression in terms of Heaviside functions:

$$f = \cos t \ (h_0 - h_\pi) + (h_\pi - h_{2\pi}) = \cos t + (1 - \cos t) h_\pi - h_{2\pi}.$$

Since $h_c(t) = h(t - c)$ the corollary gives

$$\begin{aligned} F(s) &= \frac{s}{s^2+1} + e^{-s\pi} \mathcal{L}\left\{1 - \cos(t + \pi)\right\} - \frac{e^{-2s\pi}}{s} \\ &= \frac{s}{s^2+1} + e^{-s\pi}\left(\frac{1}{s} + \frac{s}{s^2+1}\right) - \frac{e^{-2s\pi}}{s}. \end{aligned}$$

In the second line we have used the fact that $\cos(t + \pi) = -\cos t$. ◀

Exercises

Graph each of the following functions defined by means of the unit step function $h(t-c)$ and/or the on-off switches $\chi_{[a,b)}$.

1. $f(t) = 3h(t-2) - h(t-5)$
2. $f(t) = 2h(t-2) - 3h(t-3) + 4h(t-4)$
3. $f(t) = (t-1)h(t-1)$
4. $f(t) = (t-2)^2 h(t-2)$
5. $f(t) = t^2 h(t-2)$
6. $f(t) = h(t-\pi)\sin t$

7. $f(t) = h(t-\pi)\cos 2(t-\pi)$
8. $f(t) = t^2\chi_{[0,\,1)} + (1-t)\chi_{[1,\,3)} + 3\chi_{[3,\,\infty)}$

For each of the following functions $f(t)$, (a) express $f(t)$ in terms of on-off switches, (b) express $f(t)$ in terms of Heaviside functions, and (c) compute the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$.

9. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2, \\ t-2 & \text{if } 2 \le t < \infty. \end{cases}$

10. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2, \\ t & \text{if } 2 \le t < \infty. \end{cases}$

11. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2, \\ t+2 & \text{if } 2 \le t < \infty. \end{cases}$

12. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4, \\ (t-4)^2 & \text{if } 4 \le t < \infty. \end{cases}$

13. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4, \\ t^2 & \text{if } 4 \le t < \infty. \end{cases}$

14. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4, \\ t^2-4 & \text{if } 4 \le t < \infty. \end{cases}$

15. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2, \\ (t-4)^2 & \text{if } 2 \le t < \infty. \end{cases}$

16. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4, \\ e^{t-4} & \text{if } 4 \le t < \infty. \end{cases}$

17. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4, \\ e^{t} & \text{if } 4 \le t < \infty. \end{cases}$

18. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 6, \\ e^{t-4} & \text{if } 6 \le t < \infty. \end{cases}$

19. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 4, \\ te^{t} & \text{if } 4 \le t < \infty. \end{cases}$

20. $f(t) = \begin{cases} 1 & \text{if } \ 0 \le t < 4 \\ -1 & \text{if } \ 4 \le t < 5 \\ 0 & \text{if } \ 5 \le t < \infty. \end{cases}$

21. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < 2 \\ 1 & \text{if } 2 \le t < \infty. \end{cases}$

22. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < \infty. \end{cases}$

23. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ t-1 & \text{if } 1 \le t < 2 \\ t-2 & \text{if } 2 \le t < 3 \\ \vdots & . \end{cases}$

24. $f(t) = \begin{cases} 1 & \text{if } 2n \le t < 2n+1 \\ 0 & \text{if } 2n+1 \le t < 2n+2. \end{cases}$

25. $f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 2 \\ 4 & \text{if } 2 \le t < 3 \\ 7-t & \text{if } 3 \le t < \infty. \end{cases}$

26. $f(t) = \begin{cases} 1-t & \text{if } 0 \le t < 2 \\ 3-t & \text{if } 2 \le t < 4 \\ 5-t & \text{if } 4 \le t < 6 \\ \vdots & . \end{cases}$

27. $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ 3-t & \text{if } 2 \le t < 3 \\ 2(t-3) & \text{if } 3 \le t < 4 \\ 2 & \text{if } 4 \le t < \infty. \end{cases}$

4.3 The Inversion of the Laplace Transform

We now turn our attention to the inversion of the Laplace transform. In Chapter 2 we established a one-to-one correspondence between elementary functions and proper rational functions: for each proper rational function its inverse Laplace transform is a unique elementary function. For the Heaviside class the matter is complicated by our allowing discontinuity. Two functions f_1 and f_2 are said to be **essentially equal** if for each interval $[0, N)$ they are equal as functions except at possibly finitely many points.

For example, the functions

$$f_1(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 2 & \text{if } 1 \leq t < \infty \end{cases} \quad f_2(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 3 & \text{if } t = 1 \\ 2 & \text{if } 1 < t < \infty \end{cases} \quad f_3(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 2 & \text{if } 1 < t < \infty. \end{cases}$$

are essentially equal for they are equal everywhere except at $t = 1$. Two functions that are essentially equal have the same Laplace transform. This is because the Laplace transform is an integral operator and integration cannot distinguish functions that are essentially equal. The Laplace transform of f_1, f_2, and f_3 in our example above are all $\frac{1}{s} + \frac{e^{-s}}{s}$. Here is our problem: Given a transform, like $\frac{1}{s} + \frac{e^{-s}}{s}$, how do we decide what 'the' inverse Laplace transform is. It turns out that if $F(s)$ is the Laplace transform of functions $f_1, f_2 \in \mathcal{H}$ then f_1 and f_2 are essentially equal. For most practical situations it does not matter which one is chosen. However, in this text we will consistently use the one that is right continuous at each point. A function f in the Heaviside class is said to be **right continuous at a point** a if we have

$$f(a) = f(a^+) = \lim_{t \to a^+} f(t),$$

and it is **right continuous on** $[0, \infty)$ if it is right continuous at each point in $[0, \infty)$. In the example above, f_1 is right continuous while f_2 and f_3 are not. The function f_3 is, however, left continuous, using the obvious definition of left continuity. If we decide to use right continuous functions in the Heaviside class then the correspondence with its Laplace transform is one-to-one. We summarize this discussion as a theorem:

Theorem 4.3.1. *If $F(s)$ is the Laplace transform of a function in $\mathcal{H}$ then there is a unique right continuous function $f \in \mathcal{H}$ such that $\mathcal{L}\{f\} = F$. Any two functions in $\mathcal{H}$ with the same Laplace transform are essentially equal.*

Recall from our definition that h_c is right continuous. So piecewise functions written as sums of products of a continuous function and a Heaviside function are right continuous.

Example 4.3.2. Find the inverse Laplace transform of

$$F(s) = \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s-4}$$

and write it as a right continuous piecewise function.

► **Solution.** The inverse Laplace transforms of $\frac{1}{s^2}$ and $\frac{1}{s-4}$ are, respectively, t and e^{4t}. By Theorem 4.2.4 the inverse Laplace transform of $F(s)$ is

$$(t-1)h_1 + e^{4(t-3)}h_3.$$

On the interval $[0, 1)$ both $t-1$ and $e^{4(t-3)}$ are off. On the interval $[1, 3)$ only $t-1$ is on. On the interval $[3, \infty)$ both $t-1$ and $e^{4(t-3)}$ are on. Thus

$$\mathcal{L}^{-1}\{F(s)\} = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ t-1 & \text{if } 1 \leq t < 3 \\ t-1+e^{4(t-3)} & \text{if } 3 \leq t < \infty \end{cases}.$$

◀

The Laplace Transform of t^α and the Gamma function

We showed in Chapter 2 that the Laplace transform of t^n is $\frac{n!}{s^{n+1}}$, for each nonnegative integer n. One might conjecture that the Laplace transform of t^α, for α an arbitrary nonnegative real number, is given by a similar formula. Such a formula would necessarily extend the notion of 'factorial'. We define the **gamma function** by the formula

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt.$$

It can be shown that the improper integral that defines the gamma function converges as long as α is greater than 0. The following proposition, whose proof is left as an exercise, establishes the fundamental properties of the gamma function.

Proposition 4.3.3.

1. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ *(The fundamental recurrence relation)*
2. $\Gamma(1) = 1$
3. $\Gamma(n+1) = n!$

The third formula in the proposition allows us to rewrite the Laplace transform of t^n in the following way:

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}.$$

If $\alpha > -1$ we obtain

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

(Even though t^α is not in the Heaviside class for $-1 < \alpha < 0$ its Laplace transform still exists.) To establish this formula fix $\alpha > -1$. By definition

$$\mathcal{L}\left\{t^\alpha\right\} = \int_0^\infty e^{-st} t^\alpha \, dt.$$

We make the change of variable $u = st$. Then $du = sdt$ and

$$\begin{aligned} \mathcal{L}\left\{t^\alpha\right\}(s) &= \int_0^\infty e^{-u} \frac{u}{s} \frac{du}{s} \\ &= \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-u} u^\alpha \, du \\ &= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}. \end{aligned}$$

Of course, in order to actually compute the Laplace transform of some non integer positive power of t one must know the value of the gamma function for the corresponding power. For example, it is known that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. By the fundamental recurrence relation $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$. Therefore

$$\mathcal{L}\left\{t^{\frac{1}{2}}\right\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}.$$

Exercises

Compute the inverse Laplace transform of each of the following functions.

1. $\dfrac{e^{-3s}}{s-1}$
2. $\dfrac{e^{-3s}}{s^2}$
3. $\dfrac{e^{-3s}}{(s-1)^3}$
4. $\dfrac{e^{-\pi s}}{s^2+1}$
5. $\dfrac{se^{-3\pi s}}{s^2+1}$

6. $\dfrac{e^{-\pi s}}{s^2+2s+5}$

7. $\dfrac{e^{-s}}{s^2}+\dfrac{e^{-2s}}{(s-1)^3}$

8. $\dfrac{e^{-2s}}{s^2+4}$

9. $\dfrac{e^{-2s}}{s^2-4}$

10. $\dfrac{se^{-4s}}{s^2+3s+2}$

11. $\dfrac{e^{-2s}+e^{-3s}}{s^2-3s+2}$

12. $\dfrac{1-e^{-5s}}{s^2}$

13. $\dfrac{1+e^{-3s}}{s^4}$

14. $e^{-\pi s}\dfrac{2s+1}{s^2+6s+13}$

15. $(1-e^{-\pi s})\,\dfrac{2s+1}{s^2+6s+13}$

4.4 Properties of the Laplace Transform

Many of the properties of the Laplace transform that we discussed in Chapter 2 for elementary functions carry over to the Heaviside class. Their proofs are the same. These properties are summarized below.

Linearity	$\mathcal{L}\{af+bg\}=a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$.
The First Translation Principle	$\mathcal{L}\{e^{-at}f\}=\mathcal{L}\{f\}(s-a)$.
Differentiation in Transform Space	$\mathcal{L}(-tf(t))=F'(s)$ $\mathcal{L}\{(-t)^n f(t)\}=F^{(n)}(s)$.
Integration in Domain Space	$\mathcal{L}(\left\{\int_0^t f(u)\,du\right\}=\frac{F(s)}{s}$.

There are a few properties though that need some clarifications. In particular, we need to discuss the meaning of the fundamental derivative formula

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0),$$

when f is in the Heaviside class. You will recall that the derivative of an elementary function is again an elementary function. However, for the Heaviside class this is not necessarily the case. A couple of things can go wrong. First, there are examples of functions in $\mathcal{H}$ for which the derivative does not exist at any point. Second, even when the derivative exists there is no guarantee that it is back in $\mathcal{H}$. As an example, consider the function

$$f(t) = \sin e^{t^2}.$$

This function is in $\mathcal{H}$ because it is bounded (between -1 and 1) and continuous. However, its derivative is

$$f'(t) = 2te^{t^2} \cos e^{t^2},$$

which is continuous but not of exponential type. To see this recall that e^{t^2} is not of exponential type. Thus at those values of t where $\cos e^{t^2} = 1$, $|f'(t)|$ is not bounded by an exponential function and hence $f' \notin \mathcal{H}$. Therefore, in order to extend the derivative formula to $\mathcal{H}$ we must include in the hypotheses the requirement that both f and f' be in $\mathcal{H}$. Recall that for f in $\mathcal{H}$ the symbol f' is used to denote the derivative of f if f is differentiable except at a finite number of points on each interval of the form $[0, N]$.

The Laplace Transform of a Derivative

With these understandings we now have

Theorem 4.4.1. *If f is continuous and f and f' are in $\mathcal{H}$ then*

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0).$$

Proof. We begin by computing $\int_0^N e^{-st} f'(t)\, dt$. This integral requires that we consider the points where f' is discontinuous. There are only finitely many on $[0, N)$, $a_1, \ldots, a_k$, say, and we may assume $a_i < a_{i+1}$. If we et $a_0 = 0$ and $a_{k+1} = N$ then we obtain

$$\int_0^N e^{-st} f'(t)\, dt = \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} e^{-st} f'(t)\, dt,$$

and integration by parts gives

$$\begin{aligned}
\int_0^N e^{-st} f'(t)\,dt &= \sum_{i=0}^{k} \left(f(t)e^{-st}\Big|_{a_i}^{a_{i+1}} + s\int_{a_i}^{a_{i+1}} e^{-st} f(t)\,dt \right) \\
&= \sum_{i=0}^{k} (f(a_{i+1}^-)e^{-sa_{i+1}} - f(a_i^+)e^{-sa_i}) + \int_0^N e^{-st} f(t)\,dt \\
&= f(N)e^{-Ns} - f(0) + s\int_0^N e^{-st} f(t)\,dt.
\end{aligned}$$

We have used the continuity of f to make the evaluations at a_i and a_{i+1}, which allows for the collapsing sum in line 2. We now take the limit as N goes to infinity and the result follows. □

The following corollary is immediate:

Corollary 4.4.2. *If f and f' are continuous and f, f', and f'' are in $\mathcal{H}$ then*

$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0).$$

The Laplace Transform Method

The differential equations that we will solve by means of the Laplace transform are first and second order constant coefficient linear differential equations with a forcing function f in $\mathcal{H}$:

$$\begin{gathered}
y' + ay = f(t) \\
y'' + ay' + by = f(t).
\end{gathered}$$

In order to apply the Laplace transform method we will need to know that there is a solution y which is continuous in the first equation and both y and y' are continuous in the second equation. These facts were proved in Theorems 4.1.6 and 4.1.8.

We are now in a position to illustrate the Laplace transform method to solve differential equations with possibly discontinuous forcing functions f.

Example 4.4.3. Solve the following first order differential equation:

$$y' + 2y = f(t), \qquad y(0) = 1,$$

where

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ t & \text{if } 1 \le t < \infty. \end{cases}$$

▶ **Solution.** We first rewrite f in terms of Heaviside functions: $f(t) = t\,\chi_{[1,\infty)}(t) = t\,h_1(t)$. By Corollary 4.2.5 its Laplace transform is $F(s) = e^{-s}\mathcal{L}\{t+1\} = e^{-s}(\frac{1}{s^2} + \frac{1}{s}) = e^{-s}(\frac{s+1}{s^2})$. The Laplace transform of the differential equation yields

$$sY(s) - y(0) + 2Y(s) = e^{-s}(\frac{s+1}{s^2}),$$

and solving for Y gives

$$Y(s) = \frac{1}{s+2} + e^{-s}\frac{s+1}{s^2(s+2)}.$$

A partial fraction decomposition gives

$$\frac{s+1}{s^2(s+2)} = \frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2},$$

and the second translation principle (Theorem 4.2.4) gives

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s^2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s+2}\right\} \\ &= e^{-2t} + \frac{1}{4}h_1 + \frac{1}{2}(t-1)h_1 - \frac{1}{4}e^{-2(t-1)}h_1. \\ &= \begin{cases} e^{-2t} & \text{if } 0 \le t < 1 \\ e^{-2t} + \frac{1}{4}(2t-1) - \frac{1}{4}e^{-2(t-1)} & \text{if } 1 \le t < \infty. \end{cases} \end{aligned}$$

◀

We now consider a mixing problem of the type mentioned in the introduction to this chapter.

Example 4.4.4. Suppose a tank holds 10 gallons of pure water. There are two input sources of brine solution: the first source has a concentration of 2 pounds of salt per gallon while the second source has a concentration of 3 pounds of salt per gallon. The first source flows into the tank at a rate of 1 gallon per minute for 5 minutes after which it is turned off and simultaneously the second source is turned on at a rate of 1 gallon per minute. The well mixed solution flows out of the tank at a rate of 1 gallon per minute. Find the amount of salt in the tank at any time t.

▶ **Solution.** The principles we considered in Chapter 1.1 apply here:

$$y'(t) = \text{Rate in} - \text{Rate out}.$$

Recall that the input and output rates of salt are the product of the concentration of salt and the flow rates of the solution. The rate at which salt is input depends on the interval of time. For the first five minutes, source one inputs salt at a rate of 2 lbs per minute, and after that, source two inputs salt at a rate of 3 lbs per minute. Thus the input rate is represented by the function

$$f(t) = \begin{cases} 2 & \text{if } 0 \le t < 5 \\ 3 & \text{if } 5 \le t < \infty. \end{cases}$$

The rate at which salt is output is $\frac{y(t)}{10}$ lbs per minute. We therefore have the following differential equation and initial condition:

$$y' = f(t) - \frac{y(t)}{10}, \qquad y(0) = 0.$$

Rewriting f in terms of Heaviside functions gives $f = 2\chi_{[0,5)} + 3\chi_{[5,\infty)} = 2(h_0 - h_5) + 3h_5 = 2 + h_5$. Applying the Laplace transform to the differential equation and solving for $Y(s) = \mathcal{L}\{y\}(s)$ gives

$$\begin{aligned} Y(s) &= \left(\frac{1}{s+\frac{1}{10}}\right)\left(\frac{2+e^{-s}}{s}\right) \\ &= \frac{2}{(s+\frac{1}{10})s} + e^{-5s}\frac{1}{(s+\frac{1}{10})s} \\ &= \frac{20}{s} - \frac{20}{s+\frac{1}{10}} + e^{-5s}\frac{10}{s} - e^{-5s}\frac{10}{s+\frac{1}{10}}. \end{aligned}$$

Taking the inverse Laplace transform of $Y(s)$ gives

$$\begin{aligned} y(t) &= 20 - 20e^{-\frac{t}{10}} + 10h_5(t) - 10e^{-\frac{t-5}{10}}h_5(t) \\ &= \begin{cases} 20 - 20e^{-\frac{t}{10}} & \text{if } 0 \le t < 5 \\ 30 - 20e^{-\frac{t}{10}} - 10e^{-\frac{t-5}{10}} & \text{if } 5 \le t < \infty. \end{cases} \end{aligned}$$

The graph of y is given in Figure 4.13. As expected we observe that the solution is continuous, but the small kink at $t = 5$ indicates that there is a discontinuity of the derivative at this point. This occurred when the flow of the second source, which had a higher concentration of salt, was turned on. ◀

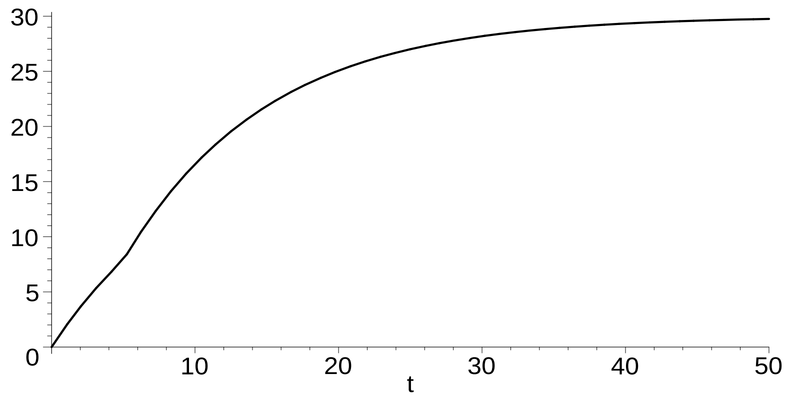


Figure 4.13: The solution to a mixing problem with discontinuous input function.

Exercises

Solve each of the following initial value problems.

1. $y' + 2y = f(t)$ where $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ -3 & \text{if } t \ge 1 \end{cases}$ $\quad y(0) = 0.$

2. $y' + 2y = f(t)$ where $f(t) = \begin{cases} -2 & \text{if } 0 \le t < 1 \\ 2 & \text{if } t \ge 1 \end{cases}$ $\quad y(0) = 0.$

3. $y' + 2y = f(t)$ where $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 2 & \text{if } 1 \le t < 3 \\ 0 & \text{if } t \ge 3 \end{cases}$ $\quad y(0) = 0.$

4. $y' + 2y = f(t)$ where $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$ $\quad y(0) = 0.$

5. $y'' + 9y = h(t-3), \quad y(0) = 0,\ y'(0) = 0.$

6. $y'' - 5y' + 4y = f(t)$ where $f(t) = \begin{cases} 1 & \text{if } 0 \le t < 5 \\ 0 & \text{if } t \ge 5 \end{cases}$ $\quad y(0) = 0,\ y'(0) = 1.$

7. $y'' + 5y' + 6y = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 2 & \text{if } 1 \le t < 3 \\ 0 & \text{if } t \ge 3 \end{cases}$ $\quad y(0) = 0,\ y'(0) = 0.$

8. $y'' + 9y = h(t-2\pi)\sin t, \quad y(0) = 1,\ y'(0) = 0.$

9. $y'' + 2y' + y = h(t-3), \quad y(0) = 0, \, y'(0) = 1.$

10. $y'' + 2y' + y = h(t-3)e^t, \quad y(0) = 0, \, y'(0) = 1.$

11. $y'' + 6y' + 5y = 1 - h(t-2) + h(t-4) + h(t-6), \quad y(0) = 0, \, y'(0) = 0.$

4.5 The Dirac Delta Function

In applications we may encounter an input into a system we wish to study that is very large in magnitude, but applied over a short period of time. Consider, for example, the following mixing problem:

Example 4.5.1. A tank holds 10 gallons of a brine solution in which each gallon contains 2 pounds of dissolved salt. An input source begins pouring fresh water into the tank at a rate of 1 gallon per minute and the thoroughly mixed solution flows out of the tank at the same rate. After 5 minutes 3 pounds of salt are poured into the tank where it instantly mixes into the solution. Find the amount of salt at any time t.

This example introduces a sudden action, namely, the sudden input of 3 pounds of salt at time $t = 5$ minutes. If we imagine that it actually takes 1 second to do this then the average rate of input of salt would be 3 lbs/ sec = 180 lbs/min. Thus we see a high magnitude in the rate of input of salt over a short interval. Moreover, the rate multiplied by the duration of input gives the total input.

More generally, if $r(t)$ represents the rate of input over a time interval $[a, b]$ then $\int_a^b r(t)\, dt$ would represent the total input over that interval. A unit input means that this integral is 1. Let $t = c \geq 0$ be fixed and let ϵ be a small positive number. Imagine a constant input rate over the interval $[c, c+\epsilon)$ and 0 elsewhere. The function $d_{c,\epsilon} = \frac{1}{\epsilon}\chi_{[c,c+\epsilon)}$ represents such an input rate with constant input $(\frac{1}{\epsilon})$ over the interval $[c, c+\epsilon)$ (c.f. section 4.2 where the on-off switch $\chi_{[a,b)}$ is discussed). The constant $\frac{1}{\epsilon}$ is chosen so that the total input is

$$\int_0^\infty d_{c,\epsilon}\, dt = \frac{1}{\epsilon}\int_c^{c+\epsilon} 1\, dt = \frac{1}{\epsilon}\epsilon = 1.$$

For example, if $\epsilon = \frac{1}{60}$ min, then $3d_{5,\epsilon}$ would represent the input of 3 lbs of salt over a 1 second interval beginning at $t = 5$.

Figure 4.14 shows the graphs of $d_{c,\epsilon}$ for a few values of ϵ. The main idea will be to take smaller and smaller values of ϵ, i.e. we want to imagine the total input being concentrated

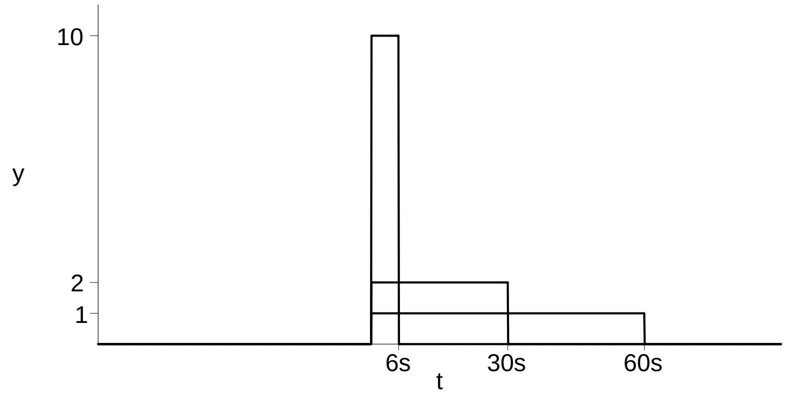


Figure 4.14: Approximation to a delta function

at the point c. Formally, we define the **Dirac delta function** by $\delta_c(t) = \lim_{\epsilon \to 0^+} d_{c,\epsilon}(t)$. Heuristically, we would like to write

$$\delta_c(t) = \begin{cases} \infty & \text{if } t = c \\ 0 & \text{elsewhere,} \end{cases}$$

with the property that $\int_0^\infty \delta_c(t)\, dt = \lim_{\epsilon \to 0} \int_0^\infty d_{c,\epsilon}\, dt = 1$. Of course, there is really no such function with this property. (Mathematically, we can make precise sense out of this idea by extending the Heaviside class to a class that includes **distributions** or **generalized functions**. We will not pursue distributions here as it will take us far beyond the introductory nature of this text.) Nevertheless, this is the idea we want to develop, at least formally. We will consider first order constant coefficient differential equations of the form

$$y' + ay = f(t)$$

where f involves the Dirac delta function δ_c. It turns out that the main problem lies in the fact that the solution is **not** continuous, so Theorem 4.4.1 does not apply. Nevertheless, we will justify that we can apply the usual Laplace transform method in a formal way to produce the desired solutions. The beauty of doing this is found in the ease in which we can work with the "Laplace transform" of δ_c.

We define the Laplace transform of δ_c by the formula:

$$\mathcal{L}\{\delta_c\} = \lim_{\epsilon \to 0} \mathcal{L}\{d_{c,\epsilon}\}.$$

Theorem 4.5.2. *The Laplace transform of δ_c is*

$$\mathcal{L}\{\delta_c\} = e^{-cs}.$$

Proof. We begin with $d_{c,\epsilon}$.

$$\begin{aligned}
\mathcal{L}\left\{d_{c,\epsilon}\right\} &= \frac{1}{\epsilon}\mathcal{L}\left\{h_c - h_{c+\epsilon}\right\} \\
&= \frac{1}{\epsilon}\left(\frac{e^{-cs} - e^{-(c+\epsilon)s}}{s}\right) \\
&= \frac{e^{-cs}}{s}\left(\frac{1 - e^{-\epsilon s}}{\epsilon}\right).
\end{aligned}$$

We now take limits as ϵ goes to 0 and use L'Hospitals rule to obtain:

$$\mathcal{L}\left\{\delta_c\right\} = \lim_{\epsilon \to 0} \mathcal{L}\left\{d_{c,\epsilon}\right\} = \frac{e^{-cs}}{s}\left(\lim_{\epsilon \to 0} \frac{1 - e^{-\epsilon s}}{\epsilon}\right) = \frac{e^{-cs}}{s} \cdot s = e^{-cs}.$$

□

We remark that when $c = 0$ we have $\mathcal{L}\left\{\delta_0\right\} = 1$. By Theorem 4.2.1 there is no Heaviside function with this property. Thus, to reiterate, even though $\mathcal{L}\left\{\delta_c\right\}$ is a function, δ_c is **not**. We will frequently write $\delta = \delta_0$. Observe that $\delta_c(t) = \delta(t - c)$.

The mixing problem from Example 4.5.1 gives rise to a first order linear differential equation involving the Dirac delta function.

► **Solution.** Let $y(t)$ be the amount of salt in the tank at time t. Then $y(0) = 20$ and y' is the difference of the input rate and the output rate. The only input of salt occurs at $t = 5$. If the salt were input over a small interval, $[5, 5 + \epsilon)$ say, then $\frac{3}{\epsilon}\chi_{[5,5+\epsilon)}$ would represent the input of 3 pounds of salt over a period of ϵ minutes. If we let ϵ go to zero then $3\delta_5$ would represent the input rate. The output rate is $y(t)/10$. We are thus led to the differential equation:

$$y' + \frac{y}{10} = 3\delta_5, \qquad y(0) = 20.$$

◄

The solution to this differential equation will fall out of the slightly more general discussion we give below.

Differential Equations of the form $y' + ay = k\delta_c$

We will present progressively four methods for solving

$$y' + ay = k\delta_c, \qquad y(0) = y_0. \qquad \star$$

The last method, the **formal Laplace Transform Method**, is the simplest method and is, in part, justified by the methods that precede it. The formal method will thereafter be used to solve equations of the form $\star$ and will work for all the problems introduced in this section. Keep in mind though that in practice a careful analysis of the limiting processes involved must be done to determine the validity of the formal Laplace Transform method.

Method 1. In our first approach we solve the equation

$$y' + ay = \frac{k}{\epsilon}\chi_{[c,c+\epsilon)}, \quad y(0) = y_0$$

and call the solution y_ϵ. We let $y(t) = \lim_{\epsilon\to 0} y_\epsilon$. Then $y(t)$ is the solution to $y' + ay = k\delta_c, \quad y(0) = y_0$. Recall from Exercise **??** the solution to

$$y' + ay = A\chi[\alpha, \beta), \quad y(0) = y_0,$$

is

$$y(t) = y_0 e^{-at} + \frac{A}{a}\begin{cases} 0 & \text{if } 0 \le t < \alpha \\ 1 - e^{-a(t-\alpha)} & \text{if } \alpha \le t < \beta \\ e^{-a(t-\beta)} - e^{-a(t-\alpha)} & \text{if } \beta \le t < \infty. \end{cases}$$

We let $A = \frac{k}{\epsilon}$, $\alpha = c$, and $\beta = c + \epsilon$ to get

$$y_\epsilon(t) = y_0 e^{-at} + \frac{k}{a\epsilon}\begin{cases} 0 & \text{if } 0 \le t < c \\ 1 - e^{-a(t-c)} & \text{if } c \le t < c+\epsilon \\ e^{-a(t-c-\epsilon)} - e^{-a(t-c)} & \text{if } c+\epsilon \le t < \infty. \end{cases}$$

The computation of $\lim_{\epsilon\to 0} y_\epsilon$ is done on each interval separately. If $0 \le t \le c$ then $y_\epsilon = y_0 e^{-at}$ is independent of ϵ and hence

$$\lim_{\epsilon\to 0} y_\epsilon(t) = y_0 e^{-at} \qquad 0 \le t \le c.$$

If $c < t < \infty$ then for ϵ small enough, $c + \epsilon < t$ and thus

$$y_\epsilon(t) = y_0 e^{-at} + \frac{k}{a\epsilon}(e^{-a(t-c-\epsilon)} - e^{-a(t-c)}) = y_0 e^{-at} + \frac{k}{a}e^{-a(t-c)}\frac{e^{a\epsilon} - 1}{\epsilon}.$$

Therefore

$$\lim_{\epsilon\to 0} y_\epsilon(t) = y_0 e^{-at} + k e^{-a(t-c)} \qquad c < t < \infty.$$

We thus obtain

$$y(t) = \begin{cases} y_0 e^{-at} & \text{if } 0 \le t \le c \\ y_0 e^{-at} + k e^{-a(t-c)} & \text{if } c < t < \infty. \end{cases}$$

In the mixing problem above the infusion of 3 pounds of salt after five minutes will instantaneously increase the amount of salt by 3; a jump discontinuity at $t = 5$. This is seen in the solution y above. At $t = c$ there is a jump discontinuity of jump k. Of course, the solution to the mixing problem is obtained by setting $a = \frac{1}{10}$, $k = 3$, $c = 5$, and $y_0 = 20$:

$$y(t) = \begin{cases} 20e^{-\frac{t}{10}} & \text{if } 0 \le t \le 5 \\ 20e^{-\frac{t}{10}} + 3e^{-\frac{t-5}{10}} & \text{if } 5 < t < \infty, \end{cases}$$

whose graph is given in Figure 4.15. We observe that $y(5^-) = 20e^{-1/2} \simeq 12.13$ and

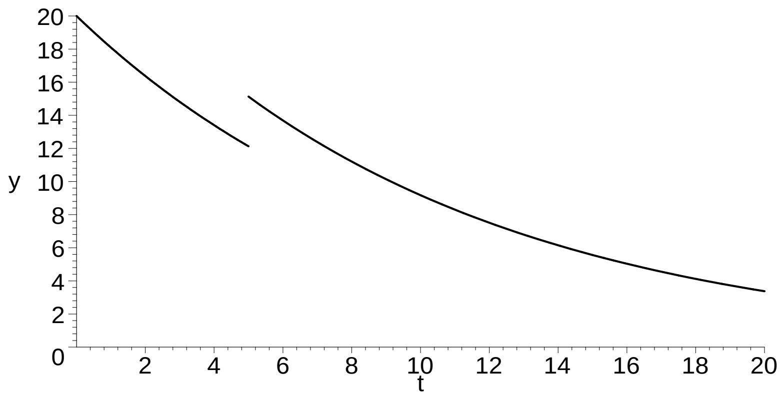


Figure 4.15: Graph of the Solution to the Mixing Problem

$y(5^+) = 20e^{-1/2} + 3 \simeq 15.13$. Also notice that $y(5^+)$ is $y(5^-)$ plus the jump 3.

Method 2. Our second approach realizes that the mixing problem stated above can be thought of as the differential equation, $y' + \frac{1}{10}y = 0$, defined on two separate intervals; (1) on the interval $[0, 5)$ with initial value $y(0) = 20$ and (2) on the interval $[5, \infty)$ where the initial value $y(5)$ is the value of the solution given in part (1) at $t = 5$, plus the jump 3. We apply this idea to our more generic initial value problem, Equation $\star$.

On the interval $[0, c)$ we solve $y' + ay = 0$ with initial value $y(0) = y_0$. The general solution is easily seen to be $y = be^{-at}$. The initial value $y(0) = y_0$ gives $b = y_0$. The solution on $[0, c)$ is thus

$$y = y_0 e^{-at}.$$

On the interval $[c, \infty)$ we solve $y' + ay = 0$ with initial value $y(c) = y_0 e^{-ac} + k$. ($y(c)$ is the value of the solution just obtained at $t = c$ plus the jump k.) Again the general solution is $y = be^{-at}$ and the initial condition implies $be^{-ac} = y_0e^{-ac} + k$. Solving for b gives $b = y_0 + ke^{ac}$. Thus

$$y = y_0e^{-at} + ke^{-a(t-c)},$$

on the interval $[c, \infty)$. Piecing these two solutions together yields

$$y = \begin{cases} y_0e^{-at} & \text{if } 0 \leq t < c \\ y_0e^{-at} + ke^{-a(t-c)} & \text{if } c \leq t < \infty \end{cases},$$

which, as it should be, is the same solution we obtained by method 1.

Method 3. In this method we want to focus on the differential equation, $y' + ay = 0$ on the entire interval $[0, \infty)$ with the a priori knowledge that there is a jump discontinuity at $t = c$. Recall from Theorem 4.4.1 that when y is continuous and both y and y' are in $\mathcal{H}$ we have the formula

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0).$$

We cannot apply this theorem as stated for y is not continuous. But if y has a single jump discontinuity at $t = c$ we can prove a slight generalization of Theorem 4.4.1.

Theorem 4.5.3. *Suppose y and y' are in $\mathcal{H}$ and y is continuous except for one jump discontinuity at $t = c$ with jump k. Then*

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) - ke^{-cs}.$$

Proof. Let $N > c$. Then integration by parts gives

$$\begin{aligned} \int_0^N e^{-st}y'(t)\,dt &= \int_0^c e^{-st}y(t)\,dt + \int_c^N e^{-st}y(t)\,dt \\ &= e^{-st}y(t)|_0^c + s\int_0^c e^{-st}y(t)\,dt + e^{-st}y(t)|_c^N + s\int_c^N e^{-st}y(t)\,dt \\ &= s\int_0^N e^{-st}y(t)\,dt + e^{-sN}y(N) - y(0) - e^{-sc}(y(c^+) - y(c^-)). \end{aligned}$$

We take the limit as N goes to infinity and obtain:

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) - ke^{-sc}.$$

□

We apply this theorem to the initial value problem

$$y' + ay = 0, \qquad y(0) = y_0$$

with the knowledge that the solution y has a jump discontinuity at $t = c$ with jump k. Apply the Laplace transform to to the differential equation to obtain:

$$sY(s) - y(0) - ke^{-ac} + aY(s) = 0.$$

Solving for Y gives

$$Y(s) = \frac{y_0}{s+a} + k\frac{e^{-as}}{s+a}.$$

Applying the inverse Laplace transform gives the solution

$$\begin{aligned} y(t) &= y_0 e^{-at} + ke^{-a(t-c)} h_c(t) \\ &= \begin{cases} y_0 e^{-at} & \text{if } 0 \le t < c \\ y_0 e^{-at} + ke^{-a(t-c)} & \text{if } c \le t < \infty. \end{cases} \end{aligned}$$

Method 4: The Formal Laplace Transform Method. We now return to the differential equation

$$y' + ay = k\delta_c, \qquad y(0) = y_0$$

and apply the Laplace transform method directly. That we can do this is partly justified by method 3 above. From Theorem 4.5.2 the Laplace transform of $k\delta_c$ is ke^{-sc}. This is precisely the term found in Theorem 4.5.3 where the assumption of a single jump discontinuity is assumed. Thus the presence of $k\delta_c$ automatically encodes the jump discontinuity in the solution. Therefore we can (formally) proceed without any advance knowledge of jump discontinuities. The Laplace transform of

$$y' + ay = k\delta_c, \qquad y(0) = y_0$$

gives

$$sY(s) - y(0) + kY(s) = ke^{-sc}$$

and one proceeds as at the end of method 3 to get

$$y(t) = \begin{cases} y_0 e^{-at} & \text{if } 0 \le t < c \\ y_0 e^{-at} + ke^{-a(t-c)} & \text{if } c \le t < \infty. \end{cases}$$

4.6 Impulse Functions

An impulsive force is a force with high magnitude introduced over a short period of time. For example, a bat hitting a ball or a spike in electricity on an electric circuit both involve impulsive forces and are best represented by the Dirac delta function. In this section we will consider the effect of the introduction of impulsive forces into such systems and how they lead to second order differential equations of the form

$$my'' + \mu y' + ky = K\delta_c(t).$$

As we will soon see the effect of an impulsive force introduces a discontinuity not in y but its derivative y'.

If $F(t)$ represents a force which is 0 outside a time interval $[a, b]$ then $\int_0^\infty F(t)\,dt = \int_a^b F(t)\,dt$ represents the total **impulse** of the force $F(t)$ over that interval. A unit impulse means that this integral is 1. If F is given by the acceleration of a constant mass then $F(t) = ma(t)$, where m is the mass and $a(t)$ is the acceleration. The total impulse

$$\int_a^b F(t)\,dt = \int_a^b ma(t)\,dt = mv(b) - mv(a)$$

represents the change of momentum. (Momentum is the product of mass and velocity). Now imagine this force is introduced over a very short period of time, or even instantaneously. As in the previous section, we could model the force by $d_{c,\epsilon} = \frac{1}{\epsilon}\chi_{[c,c+\epsilon)}$ and one would naturally be lead to the Dirac delta function to represent the instantaneous change of momentum. Since momentum is proportional to velocity we see that such impacts lead to discontinuities in the derivative y'.

Example 4.6.1. (see Chapter 3.8 for a discussion of spring-mass-dashpot systems) A spring is stretched 49 cm when a 1 kg mass is attached. The body is pulled to 10 cm below its spring-body equilibrium and released. We assume the system is frictionless. After 3 sec the mass is suddenly struck by a hammer in a downward direction with total impulse of 4 kg·m/sec. Find the motion of the mass.

▶ **Solution.** We will work in units of kg, m, and sec. Thus the spring constant k is given by $1(9.8) = k\frac{49}{100}$, so that $k = 20$. The initial conditions are given by $y(0) = .10$ and $y'(0) = 0$, and since the system is frictionless the rewritten initial value problem is

$$y'' + 20y = 4\delta_3, \qquad y(0) = .10, \quad y'(0) = 0.$$

◀

We will return to the solution of this problem after we discuss the more general second order case.

Differential Equations of the form $y'' + ay' + by = K\delta_c$

Our goal is to solve

$$y'' + ay' + by = K\delta_c, \qquad y(0) = y_0, \quad y'(0) = y_1 \qquad \star$$

using the formal Laplace transform method that we discussed in Method 4 of Section 4.5.

As we discussed above the effect of $K\delta_c$ is to introduce a single jump discontinuity in y' at $t = c$ with jump K. Therefore the solution to $(\star)$ is equivalent to solving

$$y'' + ay' + by = 0$$

with the advanced knowledge that y' has a jump discontinuity at $t = c$. If we apply Theorem 4.5.3 to y' we obtain

$$\begin{aligned}\mathcal{L}\{y''\} &= s\mathcal{L}\{y'\} - y'(0) - Ke^{-sc} \\ &= s^2 Y(s) - sy(0) - y'(0) - Ke^{-sc}\end{aligned}$$

Therefore, the Laplace transform of $y'' + ay' + by = 0$ leads to

$$(s^2 + as + b)Y(s) - sy(0) - y'(0) - Ke^{-sc} = 0.$$

On the other hand, if we (formally) proceed with the Laplace transform of Equation $(\star)$ without foreknowledge of discontinuities we obtain the equivalent equation

$$(s^2 + as + b)Y(s) - sy(0) - y'(0) = Ke^{-sc}.$$

Again, the Dirac function δ_c encodes the jump discontinuity automatically. If we proceed as usual we obtain

$$Y(s) = \frac{sy(0) + y'(0)}{s^2 + as + b} + \frac{Ke^{-sc}}{s^2 + as + b}.$$

The inversion will depend on the way the characteristic polynomial factors.

We now return to the example given above. The equation we wish to solve is

$$y'' + 20y = 4\delta_3, \qquad y(0) = .10, \quad y'(0) = 0.$$

► **Solution.** We apply the formal Laplace transform to obtain

$$Y(s) = \frac{.1s}{s^2 + 20} + \frac{e^{-3s}}{s^2 + 20}.$$

The inversion gives

$$\begin{aligned} y(t) &= \frac{1}{10}\cos(\sqrt{20}\,t) + \frac{1}{\sqrt{20}}\sin(\sqrt{20}\,(t-3))h_3(t) \\ &= \frac{1}{10}\cos(\sqrt{20}\,t) + \begin{cases} 0 & \text{if } 0 \le t < 3 \\ \frac{1}{\sqrt{20}}\sin(\sqrt{20}\,(t-3)) & \text{if } 3 \le t < \infty. \end{cases} \end{aligned}$$

Figure 4.16 gives the graph of the solution. You will note that y is continuous but the

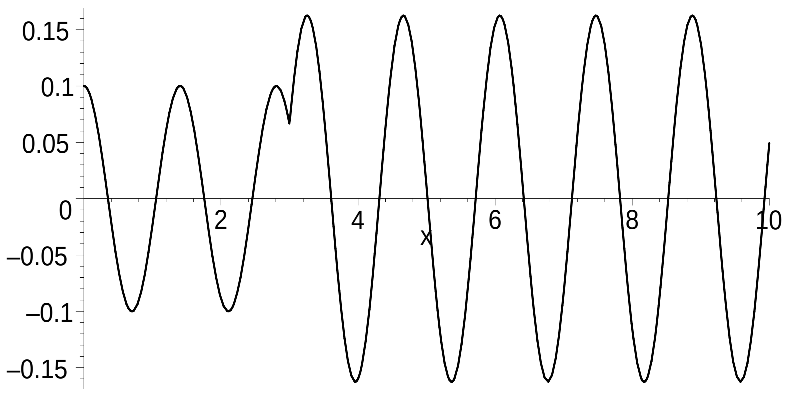


Figure 4.16: Harmonic motion with impulse function

little kink at $t = 3$ indicates the discontinuity of y'. This is precisely when the impulse to the system was delivered. ◀

Exercises

Solve each of the following initial value problems.

1. $y' + 2y = \delta_1(t), \quad y(0) = 0$
2. $y' + 2y = \delta_1(t), \quad y(0) = 1$
3. $y' + 2y = \delta_1(t) - \delta_3(t), \quad y(0) = 0$
4. $y'' + 4y = \delta_\pi(t), \quad y(0) = 0,\ y'(0) = 1$
5. $y'' + 4y = \delta_\pi(t) - \delta_{2\pi}(t), \quad y(0) = 0,\ y'(0) = 0$

6. $y'' + 4y = \delta_\pi(t) - \delta_{2\pi}(t), \quad y(0) = 1,\ y'(0) = 0$

7. $y'' + 4y' + 4y = 3\delta_1(t), \quad y(0) = 0,\ y'(0) = 0$

8. $y'' + 4y' + 4y = 3\delta_1(t), \quad y(0) = -1,\ y'(0) = 3$

9. $y'' + 4y' + 5y = 3\delta_1(t), \quad y(0) = 0,\ y'(0) = 0$

10. $y'' + 4y' + 5y = 3\delta_1(t), \quad y(0) = -1,\ y'(0) = 3$

11. $y'' + 4y' + 20y = \delta_\pi(t) - \delta_{2\pi}(t), \quad y(0) = 1,\ y'(0) = 0$

12. $y'' - 4y' - 5y = 2e^{-t} + \delta_3(t), \quad y(0) = 0,\ y'(0) = 0$

4.7 Periodic Functions

In modelling mechanical and other systems it frequently happens that the forcing function repeats over time. Periodic functions best model such repetition.

A function f defined on $[0, \infty)$ is said to be **periodic** if there is a positive number p such that $f(t + p) = f(t)$ for all t in the domain of f. We say p is **a period** of f. If $p > 0$ is a period of f and there is no smaller period then we say p is the **fundamental period** of f although we will usually just say **the period**. The interval $[0, p)$ is called the **fundamental interval**. If there is no such smallest positive p for a periodic function then the period is defined to be 0. The constant function $f(t) = 1$ is an example of a periodic function with period 0. The sine function is periodic with period 2π: $\sin(t + 2\pi) = \sin(t)$. Knowing the sine on the interval $[0, 2\pi)$ implies knowledge of the function everywhere. Similarly, if we know f is periodic with period $p > 0$ and we know the function on the fundamental interval then we know the function everywhere. Figure 4.17 illustrates this point.

The Sawtooth Function

A particularly useful periodic function is the **sawtooth** function. With it we can express other periodic functions simply by composition. Let $p > 0$. The saw tooth function is

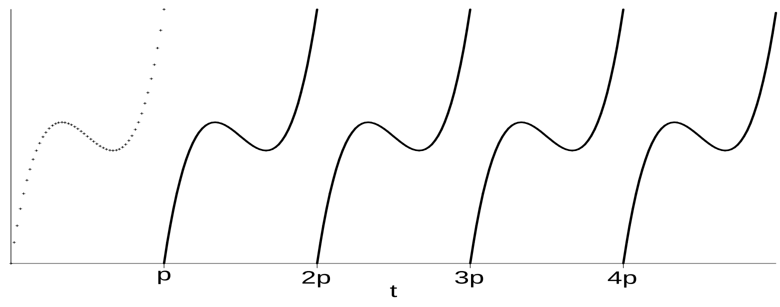


Figure 4.17: An example of a periodic function with period p. Notice how the interval $[0, p)$ determines the function everywhere.

given by

$$< t >_p = \begin{cases} t & \text{if } 0 \le t < p \\ t - p & \text{if } p \le t < 2p \\ t - 2p & \text{if } 2p \le t < 3p \\ \vdots \end{cases}.$$

It is periodic with period p. Its graph is given in Figure 4.18.

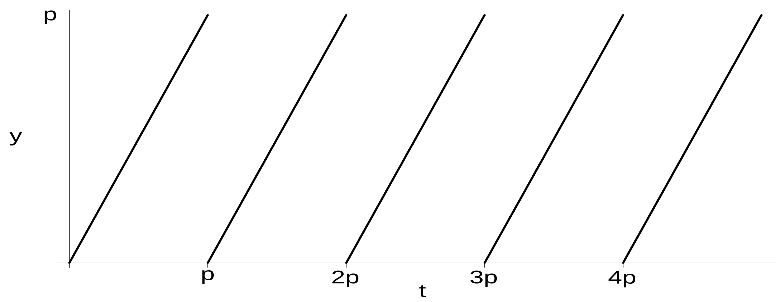


Figure 4.18: The Sawtooth Function $< t >_p$ with period p

The sawtooth function $< t >_p$ is obtained by extending the function $y = t$ on the interval $[0, p)$ periodically to $[0, \infty)$. More generally, given a function f defined on the interval $[0, p)$, we can extend it periodically to $[0, \infty)$ by the formula

$$\begin{cases} f(t) & \text{if } 0 \le t < p \\ f(t-p) & \text{if } p \le t < 2p \\ f(t-2p) & \text{if } 2p \le t < 3p \\ \vdots & . \end{cases}.$$

This complicated piecewise definition can be expressed simply by the composition of f and $< t >_p$:

$$f(< t >_p).$$

For example, Figure 4.19 is the graph of $y = \sin(< t >_\pi)$. This function, which is periodic with period π, is known as the **rectified sine wave**.

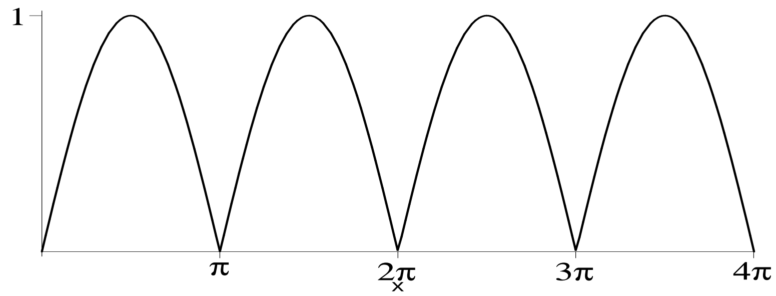


Figure 4.19: The Rectified Sine Wave: $\sin(< t >_\pi)$

The Staircase Function

Another function that will be particularly useful is the **staircase** function. For $p > 0$ it is defined as follows:

$$[t]_p = \begin{cases} 0 & \text{if } t \in [0, p) \\ p & \text{if } t \in [p, 2p) \\ 2p & \text{if } t \in [2p, 3p) \\ \vdots & \end{cases}.$$

Its graph is given in Figure 4.20. The staircase function is **not** periodic. It is useful in expressing piecewise functions that are like steps on intervals of length p. For example, if f is a function on $[0, \infty)$ then $f([t]_p)$ is a function whose value on $[np, (n + 1)p)$ is the constant $f(np)$. Figure 4.21 illustrates this idea with the function $f(t) = 1 - e^{-t}$ and $p = 0.5$.

Observe that the staircase function and the sawtooth function are related by

$$< t >_p = t - [t]_p.$$

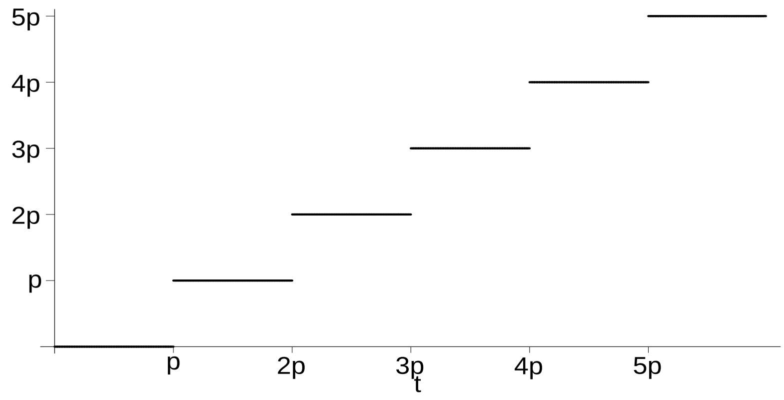


Figure 4.20: The Staircase Function: $[t]_p$

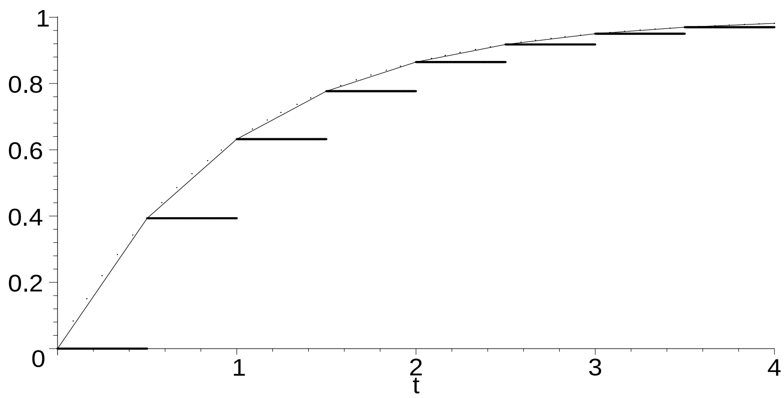


Figure 4.21: The graph of $1 - e^{-t}$ and $1 - e^{-[t]_{.5}}$

The Laplace Transform of Periodic Functions

Not surprisingly, the formula for the Laplace transform of a periodic function is determined by the fundamental interval.

Theorem 4.7.1. *Let f be a periodic function in $\mathcal{H}$ and $p > 0$ a period of f. Then*

$$\mathcal{L}\{f\}(s) = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t)\, dt.$$

Proof.

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^p e^{-st} f(t)\,dt + \int_p^\infty e^{-st} f(t)\,dt\end{aligned}$$

However, the change of variables $t \to t+p$ in the second integral and the periodicity of f gives

$$\begin{aligned}\int_p^\infty e^{-st} f(t)\,dt &= \int_0^\infty e^{-s(t+p)} f(t+p)\,dt \\ &= e^{-sp}\int_0^\infty e^{-st} f(t)\,dt \\ &= e^{-sp}\mathcal{L}\{f\}(s).\end{aligned}$$

Therefore

$$\mathcal{L}\{f\}(s) = \int_0^p e^{-st} f(t)\,dt + e^{-sp}\mathcal{L}\{f\}(s).$$

Solving for $\mathcal{L}\{f\}$ gives the desired result. □

Example 4.7.2. Find the Laplace transform of the **square-wave** function sw_c given by

$$\mathrm{sw}_c(t) = \begin{cases} 1 & \text{if } t \in [2nc, (2n+1)c) \\ 0 & \text{if } t \in [(2n+1)c, (2n+2)c) \end{cases} \quad \text{for each integer } n.$$

► **Solution.** The square-wave function sw_c is periodic with period $2c$. Its graph is given in Figure 4.22 and, by Theorem 4.7.1, its Laplace transform is

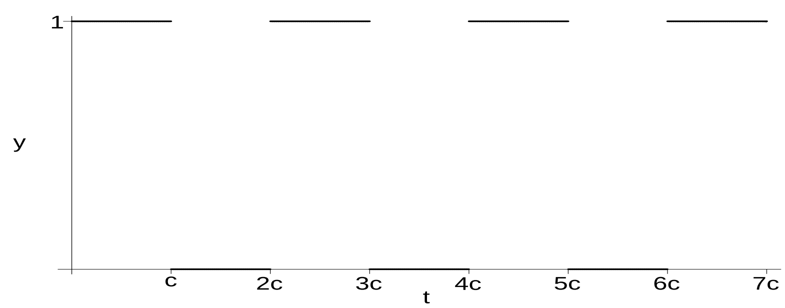


Figure 4.22: The graph of the square wave function sw_c

$$\begin{aligned}
\mathcal{L}\{\mathrm{sw}_c\}(s) &= \frac{1}{1-e^{-2cs}}\int_0^{2c} e^{-st}\,\mathrm{sw}_c(t)\,dt \\
&= \frac{1}{1-e^{-2cs}}\int_0^{c} e^{-st}\,dt \\
&= \frac{1}{1-(e^{-sc})^2}\frac{1-e^{-sc}}{s} \\
&= \frac{1}{1+e^{-sc}}\frac{1}{s}.
\end{aligned}$$

◀

Example 4.7.3. Find the Laplace transform of the sawtooth function $< t >_p$.

▶ **Solution.** Since the sawtooth function is periodic with period p and since $< t >_p = t$ for $0 \le t < p$, Theorem 4.7.1 gives

$$\mathcal{L}\{< t >_p\}(s) = \frac{1}{1-e^{-sp}}\int_0^p e^{-st}t\,dt.$$

Integration by parts gives

$$\int_0^p e^{-st}t\,dt = \frac{te^{-st}}{-s}|_0^p - \frac{1}{-s}\int_0^p e^{-st}\,dt = -\frac{pe^{-sp}}{s} - \frac{1}{s^2}e^{-st}|_0^p = -\frac{pe^{-sp}}{s} - \frac{e^{-sp}-1}{s^2}.$$

With a little algebra we obtain

$$\mathcal{L}\{< t >_p\}(s) = \frac{1}{s^2}(1 - \frac{spe^{-sp}}{1-e^{-sp}}).$$

◀

As mentioned above it frequently happens that we build periodic functions by restricting a given function f to the interval $[0, p)$ and then extending it to be periodic with period p: $f(< t >_p)$. Suppose now that $f \in \mathcal{H}$. We can then express the Laplace transform of $f(< t >_p)$ in terms of the Laplace transform of f. The following corollary expresses this relationship and simplifies unnecessary calculations like the integration by parts that we did in the previous example.

Corollary 4.7.4. *Let $p > 0$ Suppose $f \in \mathcal{H}$. Then*

$$\mathcal{L}\{f(< t >_p)\}(s) = \frac{1}{1-e^{-sp}}\mathcal{L}\{f - fh_p\}.$$

Proof. The function $f - fh_p = f(1 - h_p)$ is the same as f on the interval $[0, p)$ and 0 on the interval $[p, \infty)$.. Therefore

$$\int_0^p e^{-st} f(t)\, dt = \int_0^\infty e^{-st}(f(t) - f(t)h_p(t))\, dt = \mathcal{L}\{f - fh_p\}.$$

The result now follows from Theorem 4.7.1. □

Let's return to the sawtooth function in Example 4.7.3 and see how Corollary 4.7.4 simplifies the calculation of its Laplace transform.

$$\begin{aligned}
\mathcal{L}\{< t >_p\}(s) &= \frac{1}{1 - e^{-sp}} \mathcal{L}\{t - th_p\} \\
&= \frac{1}{1 - e^{-sp}} \left(\frac{1}{s^2} - e^{-sp} \mathcal{L}\{t + p\} \right) \\
&= \frac{1}{1 - e^{-sp}} \left(\frac{1}{s^2} - e^{-sp} \frac{1 + sp}{s^2} \right) \\
&= \frac{1}{s^2} \left(1 - \frac{spe^{-sp}}{1 - e^{-sp}} \right).
\end{aligned}$$

The last line requires a few algebraic steps.

Example 4.7.5. Find the Laplace transform of the rectified sine wave $\sin(< t >_\pi)$. See Figure 4.19.

▶ **Solution.** Corollary 4.7.4 gives

$$\begin{aligned}
\mathcal{L}\{\sin(< t >_\pi)\} &= \frac{1}{1 - e^{-\pi s}} \mathcal{L}\{\sin t - \sin t\ h_\pi(t)\} \\
&= \frac{1}{1 - e^{-\pi s}} \left(\frac{1}{s^2 + 1} - e^{-\pi s} \mathcal{L}\{\sin(t + \pi)\} \right) \\
&= \frac{1}{1 - e^{-\pi s}} \left(\frac{1 + e^{-\pi s}}{s^2 + 1} \right),
\end{aligned}$$

where we use the fact that $\sin(t + \pi) = -\sin(t)$. ◀

The inverse Laplace transform

The inverse Laplace transform of functions of the form

$$\frac{1}{1-e^{-sp}}F(s)$$

is not always a straightforward matter to find unless, of course, $F(s)$ is of the form $\mathcal{L}\{f - fh_p\}$ so that Corollary 4.7.4 can be used. Usually though this is not the case. Let r be a fixed real or complex number. Recall that the geometric series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

converges to $\frac{1}{1-r}$ when $|r| < 1$. Since $e^{-sp} < 1$ for $s > 0$ we can write

$$\frac{1}{1-e^{-sp}} = \sum_{n=0}^{\infty} e^{-snp}$$

and therefore

$$\frac{1}{1-e^{-sp}}F(s) = \sum_{n=0}^{\infty} e^{-snp}F(s).$$

If $f = \mathcal{L}^{-1}\{F\}$ then a termwise computation gives

$$\mathcal{L}^{-1}\left\{\frac{1}{1-e^{-sp}}F(s)\right\} = \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left\{e^{-snp}F(s)\right\} = \sum_{n=0}^{\infty} f(t-np)h_{np}(t).$$

On an interval of the form $[Np, (N+1)p)$ the function h_{np} is 1 for $n = 0, \ldots, N$ and 0 otherwise. We thus obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{1-e^{-sp}}F(s)\right\} = \sum_{N=0}^{\infty}\left(\sum_{n=0}^{N} f(t-np)\right)\chi_{[Np,(N+1)p)}.$$

A similar argument gives

$$\mathcal{L}^{-1}\left\{\frac{1}{1+e^{-sp}}F(s)\right\} = \sum_{N=0}^{\infty}\left(\sum_{n=0}^{N}(-1)^n f(t-np)\right)\chi_{[Np,(N+1)p)}.$$

For reference we record these results in the following theorem:

Theorem 4.7.6. *Let $p > 0$ and suppose $\mathcal{L}\{f(t)\} = F(s)$. Then*

1. $\mathcal{L}^{-1}\left\{\frac{1}{1-e^{-sp}}F(s)\right\} = \sum_{N=0}^{\infty}\left(\sum_{n=0}^{N} f(t-np)\right)\chi_{[Np,(N+1)p)}$.

2. $\mathcal{L}^{-1}\left\{\frac{1}{1+e^{-sp}}F(s)\right\} = \sum_{N=0}^{\infty}\left(\sum_{n=0}^{N}(-1)^n f(t-np)\right)\chi_{[Np,(N+1)p)}$.

Example 4.7.7. Find the inverse Laplace transform of

$$\frac{1}{(1-e^{-2s})s}.$$

▶ **Solution.** If $f(t) = 1$ then $F(s) = \frac{1}{s}$ is its Laplace transform. We thus have

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(1-e^{-2s})s}\right\} &= \sum_{N=0}^{\infty}\left(\sum_{n=0}^{N} f(t-2n)\right)\chi_{[2N,2(N+1))} \\
&= \sum_{N=0}^{\infty}(N+1)\chi_{[2N,2(N+1))} \\
&= 1 + \frac{1}{2}\sum_{N=0}^{\infty} 2N\chi_{[2N,2(N+1))} \\
&= 1 + \frac{1}{2}[t]_2.
\end{aligned}$$

◀

Mixing Problems with Periodic Input

We now turn our attention to two examples. Both are mixing problems with periodic input functions.

Example 4.7.8. Suppose a tank contains 10 gallons of pure water. Two input sources alternately flow into the tank for 1 minute intervals. The first input source is a brine solution with concentration 1 pound salt per gallon and flows (when on) at a rate of 5 gallons per minute. The second input source is pure water and flows (when on) at a rate of 5 gallons per minute. The tank has a drain with a constant outflow of 5 gallons per minute. Let $y(t)$ denote the total amount of salt at time t. Find $y(t)$ and for large values of t determine how $y(t)$ fluctuates.

▶ **Solution.** The input rate of salt is given piecewise by the formula

$$\begin{cases} 5 & \text{if } 2n \le t < 2n+1) \\ 0 & \text{if } 2n+1 \le t < 2n+2 \end{cases} = 5\,\mathrm{sw}_1(t).$$

The output rate is given by

$$\frac{y(t)}{10} \cdot 5.$$

This leads to the first order differential equation

$$y' + \frac{1}{2}y = 5\,\mathrm{sw}_1(t) \qquad y(0) = 0.$$

A calculation using Example 4.7.2 gives that the Laplace transform is

$$Y(s) = 5\frac{1}{1+e^{-s}}\,\frac{1}{s(s+\frac{1}{2})},$$

and a partial fraction decomposition gives

$$Y(s) = 10\frac{1}{1+e^{-s}}\,\frac{1}{s} - 10\frac{1}{1+e^{-s}}\,\frac{1}{s+\frac{1}{2}}.$$

Now apply the inverse Laplace transform. By Theorem 4.7.6 the inverse Laplace transform of the first expression is

$$10\sum_{N=0}^{\infty}\sum_{n=0}^{N}(-1)^n \chi_{[N,N+1)} = 10\sum_{N=0}^{\infty}\chi_{[2N,2N+1)} = 10\,\mathrm{sw}_1(t).$$

By Theorem 4.7.6 the inverse Laplace transform of the second expression is

$$\begin{aligned} 10\sum_{N=0}^{\infty}\sum_{n=0}^{N}(-1)^n e^{-\frac{1}{2}(t-n)}\chi_{[N,N+1)} &= 10e^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\sum_{n=0}^{N}(-e^{\frac{1}{2}})^n\chi_{[N,N+1)} \\ &= 10e^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\frac{1-(-e^{\frac{1}{2}})^{N+1}}{1+e^{\frac{1}{2}}}\chi_{[N,N+1)} \\ &= \frac{10e^{-\frac{1}{2}t}}{1+e^{\frac{1}{2}}}\begin{cases} 1+e^{\frac{N+1}{2}} & \text{if } t\in[N,N+1) \text{ (N even)} \\ 1-e^{\frac{N+1}{2}} & \text{if } t\in[N,N+1) \text{ (N odd)} \end{cases}. \end{aligned}$$

Finally, we put these two expression together to get our solution

$$y(t) = 10\,\mathrm{sw}_1(t) - \frac{10e^{-\frac{1}{2}t}}{1+e^{\frac{1}{2}}}\begin{cases} 1+e^{\frac{N+1}{2}} & \text{if } t\in[N,N+1) \text{ (N even)} \\ 1-e^{\frac{N+1}{2}} & \text{if } t\in[N,N+1) \text{ (N odd)}\end{cases} \tag{1}$$

$$= \begin{cases} 10-10\frac{e^{-\frac{1}{2}t}+e^{\frac{-t+N+1}{2}}}{1+e^{\frac{1}{2}}} & \text{if } t\in[N,N+1) \text{ (N even)} \\ -10\frac{e^{-\frac{1}{2}t}-e^{\frac{-t+N+1}{2}}}{1+e^{\frac{1}{2}}} & \text{if } t\in[N,N+1) \text{ (N odd)}\end{cases}.$$

The graph of $y(t)$, obtained with the help of a computer, is presented in Figure 4.23.

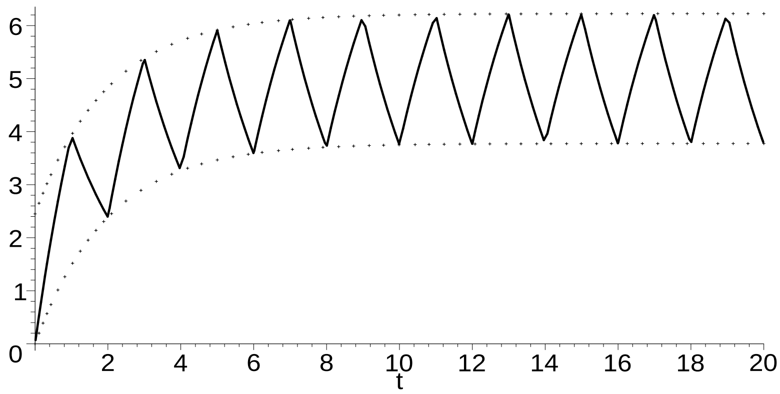


Figure 4.23: A mixing problem with square wave input function.

The solution is sandwiched in between a lower and upper curve. The lower curve, $l(t)$, is obtained by setting $t = m$ to be an even integer in the formula for the solution and then continuing it to all reals. We obtain

$$l(m) = 10 - 10\frac{e^{-\frac{1}{2}m}+e^{\frac{-m+m+1}{2}}}{1+e^{\frac{1}{2}}} = 10 - 10\frac{e^{-\frac{1}{2}m}+e^{\frac{1}{2}}}{1+e^{\frac{1}{2}}}$$

and thus

$$l(t) = 10 - 10\frac{e^{-\frac{1}{2}t}+e^{\frac{1}{2}}}{1+e^{\frac{1}{2}}}$$

In a similar way, the upper curve, $u(t)$, is obtained by setting $t = m^-$ to be an odd integer and continuing to all reals. We obtain

$$u(t) = -10\frac{e^{-\frac{1}{2}t}-e^{\frac{1}{2}}}{1+e^{\frac{1}{2}}}.$$

An easy calculation gives

$$\lim_{t\to\infty} l(t) = 10 - \frac{10e^{\frac{1}{2}}}{1+e^{\frac{1}{2}}} \simeq 3.78 \quad \text{and } \lim_{t\to\infty} u(t) = \frac{10e^{\frac{1}{2}}}{1+e^{\frac{1}{2}}} \simeq 6.22.$$

This means that the salt fluctuation in the tank varies between 3.78 and 6.22 pounds for large values of t. ◀

In practice it is not always possible to know the input function, $f(t)$, precisely. Suppose though that it is known that f is periodic with period p. Then the total input on all intervals of the form $[np,(n+1)p)$ is $\int_{np}^{(n+1)p} f(t)\,dt = h$, a constant. On the interval $[0,p)$ we could model the input with a Dirac delta function concentrated at a point, c say, and then extend it periodically. We would then obtain a sum of Dirac delta functions of the form

$$a(t) = h(\delta_c + \delta_{c+p} + \delta_{c+2p} + \cdots)$$

that may adequately represent the input for the system we are trying to model. Additional information may justify distributing the total input over two or more points in the interval and extend periodically. Whatever choices are made the solution will need to be analyzed in the light of empirical data known about the system. Consider the example above. Suppose that it is known that the input is periodic with period 2 and total input 5 on the fundamental interval. Suppose additionally that you are told that the distribution of the input of salt is on the first half of each interval. We might be led to try to model the input on $[0,2)$ by $\frac{5}{2}\delta_0 + \frac{5}{2}\delta_1$ and then extend periodically to obtain

$$a(t) = \frac{5}{2}\sum_{n=0}^{\infty} \delta_n.$$

Of course, the solution modelled by the input function $a(t)$ will differ from the actual solution. What is true though is that both exhibit similar long term behavior. This can be observed in the following example.

Example 4.7.9. Suppose a tank contains 10 gallons of pure water. Pure water flows into the tank at a rate of 5 gallons per minute. The tank has a drain with a constant outflow of 5 gallons per minute. Suppose $\frac{5}{2}$ pounds of salt is put in the tank each minute whereupon it instantly and uniformly dissolves. Assume the level of fluid in the tank is always 10 gallons. Let $y(t)$ denote the total amount of salt at time t. Find $y(t)$ and for large values of t determine how $y(t)$ fluctuates.

▶ **Solution.** As discussed above the input function is $\frac{5}{2}\sum_{n=1}^{\infty}\delta_n$ and therefore the differential equation that models this system is

$$y' + \frac{1}{2}y = \frac{5}{2}\sum_{n=1}^{\infty}\delta_n, \quad y(0) = 0.$$

The Laplace transform leads to

$$Y(s) = \frac{5}{2}\sum_{n=0}^{\infty} e^{-sn}\frac{1}{s+\frac{1}{2}},$$

and inverting the Laplace transform gives

$$\begin{aligned} y(t) &= \frac{5}{2}\sum_{n=0}^{\infty} e^{-\frac{1}{2}(t-n)}h_n(t) \\ &= \frac{5}{2}e^{-\frac{1}{2}t}\sum_{n=0}^{\infty}(e^{\frac{1}{2}})^n h_n(t) \\ &= \frac{5}{2}e^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\left(\sum_{n=0}^{N}(e^{\frac{1}{2}})^n\right)\chi_{[N,N+1)} \\ &= \frac{5}{2}e^{-\frac{1}{2}t}\sum_{N=0}^{\infty}\frac{1-e^{\frac{N+1}{2}}}{1-e^{\frac{1}{2}}}\chi_{[N,N+1)} \\ &= \frac{5(e^{-\frac{1}{2}t}-e^{-\frac{1}{2}(t-[t]-1)})}{2(1-e^{\frac{1}{2}})}. \end{aligned}$$

The graph of this equation is given in Figure 4.24. The solution is sandwiched in between

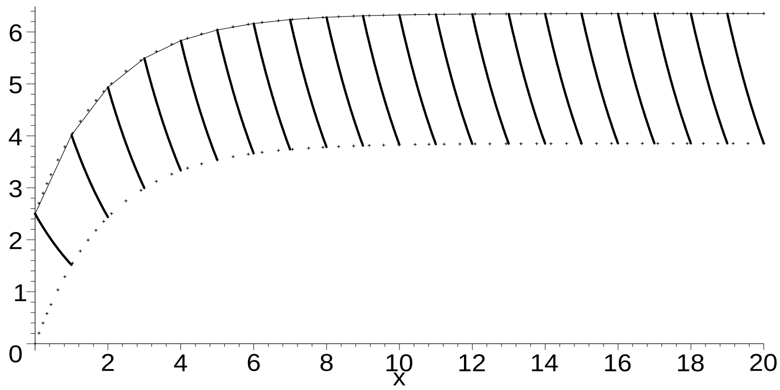


Figure 4.24: A mixing problem with a periodic Dirac delta function: The solution to the differential equation $y' + \frac{1}{2}y = \frac{5}{2}\sum_{n=1}^{\infty}\delta_n \quad y(0) = 0$.

a lower and upper curve. The lower curve, $l(t)$, is obtained by setting $t = m$ to be an integer in the formula for the solution and then continuing it to all reals. We obtain

$$l(m) = \frac{5}{2(1-e^{-\frac{1}{2}})}(e^{-\frac{m}{2}} - e^{\frac{-m+m+1}{2}}) = \frac{5}{2(1-e^{-\frac{1}{2}})}(e^{-\frac{m}{2}} - e^{\frac{1}{2}})$$

and thus

$$l(t) = \frac{5}{2(1 - e^{-\frac{1}{2}})}(e^{-\frac{t}{2}} - e^{\frac{1}{2}})$$

In a similar way, the upper curve, $u(t)$, is obtained by setting $t = (m+1)^-$ (an integer slightly less than $m+1$) and continuing to all reals. We obtain

$$u(t) = \frac{5}{2(1 - e^{-\frac{1}{2}})}(e^{-\frac{t}{2}} - 1)$$

An easy calculation gives

$$\lim_{t\to\infty} l(t) = \frac{-5e^{\frac{1}{2}}}{2(1-e^{\frac{1}{2}})} \simeq 3.85 \quad \text{and } \lim_{t\to\infty} u(t) = \frac{-5}{2(1-e^{\frac{1}{2}})} \simeq 6.35.$$

This means that the salt fluctuation in the tank varies between 3.85 and 6.35 pounds for large values of t. ◀

A comparison of the solutions in these examples reveals similar long term behavior in the fluctuation of the salt content in the tank. Remember though that each problem that is modelled must be weighed against hard empirical data to determine if the model is appropriate or not. Also, we could have modelled the instantaneous input by assuming the input was concentrated at a single point, rather than two points. The results are not as favorable. These other possibilities are explored in the exercises.

4.8 Undamped Motion with Periodic Input

In Section 3.7 we discussed various kinds of harmonic motion that can result from solutions to the differential equation

$$ay'' + by' + cy = f(t).$$

Undamped motion led to the differential equation

$$ay'' + cy = f(t). \tag{1}$$

In particular, we explored the case where $f(t) = F_0 \cos \omega t$ and were led to the solution

$$y(t) = \begin{cases} \dfrac{F_0}{a(\beta^2 - \omega^2)}(\cos \omega t - \cos \beta t) & \text{if } \beta \neq \omega \\ \\ \dfrac{F_0}{2a\omega} t \sin \omega t & \text{if } \beta = \omega, \end{cases} \tag{2}$$

where $\beta = \sqrt{\frac{c}{a}}$. The case where $\beta \neq \omega$ gave rise to the notion of beats, while the case $\beta = \omega$ gave us resonance. Since $\cos \omega t$ is periodic the system that led to Equation 1 is an example of **undamped motion with periodic input**. In this section we will explore this phenomenon with two further examples: a square wave periodic function, sw_c and a periodic impulse function, $\sum_{n=0}^{\infty} \delta_{nc}$. Both examples are algebraically tedious, so you will be asked to fill in some of the algebraic details in the exercises. To simplify the notation we will rewrite Equation (1) as

$$y'' + \beta^2 y = g(t)$$

and assume $y(0) = y'(0) = 0$.

Undamped Motion with square wave forcing function

Example 4.8.1. A constant force of r units for c units of time is applied to a mass-spring system with no damping force that is initially at rest. The force is then released for c units of time. This on-off force is extended periodically to give a periodic forcing function with period $2c$. Describe the motion of the mass.

▶ **Solution.** The differential equation which describes this system is

$$y'' + \beta^2 y = r\,\mathrm{sw}_c(t), \quad y(0) = 0,\ y'(0) = 0 \tag{3}$$

where sw_c is the square wave function with period $2c$ and β^2 is the spring constant. By Example 4.7.2 the Laplace transform leads to the equation

$$\begin{aligned} Y(s) &= r\frac{1}{1+e^{-sc}}\frac{1}{s(s^2+\beta^2)} = \frac{r}{\beta^2}\frac{1}{1+e^{-sc}}\left(\frac{1}{s} - \frac{s}{s^2+\beta^2}\right) \\ &= \frac{r}{\beta^2}\frac{1}{1+e^{-sc}}\frac{1}{s} - \frac{r}{\beta^2}\frac{1}{1+e^{-sc}}\frac{s}{s^2+\beta^2} \end{aligned} \tag{4}$$

Let

$$F_1(s) = \frac{r}{\beta^2}\frac{1}{1+e^{-sc}}\frac{1}{s} \quad \text{and} \quad F_2(s) = \frac{r}{\beta^2}\frac{1}{1+e^{-sc}}\frac{s}{s^2+\beta^2}.$$

Again, by Example 4.7.2 we have

$$f_1(t) = \frac{r}{\beta^2}\,\mathrm{sw}_c(t). \tag{5}$$

By Theorem 4.7.6 we have

$$f_2(t) = \frac{r}{\beta^2} \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} (-1)^n \cos(\beta t - n\beta c) \right) \chi_{[Nc,(N+1)c)}. \tag{6}$$

We consider two cases.

βc is not an odd multiple of π

Lemma 4.8.2. *Suppose v is not an odd multiple of π and let $\alpha = \frac{\sin(v)}{1+\cos(v)}$. Then*

1. $\sum_{n=0}^{N} (-1)^n \cos(u+nv) = \frac{1}{2}\left(\cos u + \alpha \sin u + (-1)^N (\cos(u+Nv) - \alpha \sin(u+Nv)\right)$
2. $\sum_{n=0}^{N} (-1)^n \sin(u+nv) = \frac{1}{2}\left(\sin u - \alpha \cos(u) + (-1)^N (\sin(u+Nv) + \alpha \cos(u+Nv)\right)$.

Proof. The proof of the lemma is left as an exercise. □

Let $u = \beta t$ and $v = -\beta c$. Then $\alpha = \frac{-\sin(\beta c)}{1+\cos(\beta c)}$. In this case we can apply part (1) of the lemma to Equation (6) to get

$$\begin{aligned} f_2(t) &= \frac{r}{2\beta^2} \sum_{N=0}^{\infty} \left(\cos \beta t + \alpha \sin \beta t + (-1)^N (\cos \beta(t - Nc) - \alpha \sin \beta(t - Nc)\right) \chi_{[Nc,N+1)c} \\ &= \frac{r}{2\beta^2}(\cos \beta t + \alpha \sin \beta t) + \frac{r}{2\beta^2}(-1)^{[t/c]_1}(\cos \beta < t >_c - \alpha \sin \beta < t >_c). \end{aligned} \tag{7}$$

Let

$$\begin{aligned} y_1(t) &= \frac{r}{\beta^2} \operatorname{sw}_c(t) - \frac{r}{2\beta^2}(-1)^{[t/c]_1}(\cos \beta < t >_c - \alpha \sin \beta < t >_c) \\ &= \frac{r}{2\beta^2}\left(2 \operatorname{sw}_c(t) - (-1)^{[t/c]_1}(\cos \beta < t >_c - \alpha \sin \beta < t >_c)\right) \end{aligned}$$

and

$$y_2(t) = -\frac{r}{2\beta^2}(\cos \beta t + \alpha \sin \beta t).$$

Then

$$\begin{aligned} y(t) &= f_1(t) - f_2(t) = y_1(t) + y_2(t) \\ &= \frac{r}{2\beta^2}\left(2 \operatorname{sw}_c(t) - (-1)^{[t/c]_1}(\cos \beta < t >_c - \alpha \sin \beta < t >_c)\right) \\ &\quad - \frac{r}{2\beta^2}(\cos \beta t + \alpha \sin \beta t). \end{aligned} \tag{8}$$

A quick check shows that y_1 is periodic with period $2c$ and y_2 is periodic with period $\frac{2\pi}{\beta}$. Clearly y_2 is continuous and since the solution $y(t)$ is continuous by Theorem 4.1.8, so is y_1. The following lemma will help us determine when y is a periodic solution.

Lemma 4.8.3. *Suppose g_1 and g_2 are continuous periodic functions with periods $p_1 > 0$ and $p_2 > 0$, respectively. Then $g_1 + g_2$ is periodic if and only if $\frac{p_1}{p_2}$ is a rational number.*

Proof. If $\frac{p_1}{p_2} = \frac{m}{n}$ is rational then $np_1 = mp_2$ is a common period of g_1 and g_2 and hence is a period of $g_1 + g_2$. It follows that $g_1 + g_2$ is periodic. The opposite implication, namely, that the periodicity of $g_1 + g_2$ implies $\frac{p_1}{p_2}$ is rational, is a nontrivial fact. We do not include a proof. □

Using this lemma we can determine precisely when the solution $y = y_1 + y_2$ is periodic. Namely, y is periodic precisely when $\frac{2c}{2\pi/\beta} = \frac{c\beta}{\pi}$ is rational. Consider the following illustrative example. Set $r = 2$, $c = \frac{3\pi}{2}$, and $\beta = 1$. Then $\alpha = 1$ and

$$y(t) = 2\,\mathrm{sw}_c(t) - (-1)^{[t/c]_1}(\cos < t >_c - \sin < t >_c) - (\cos t + \sin t). \tag{9}$$

This function is graphed simultaneously with the forcing function in Figure 4.25. The

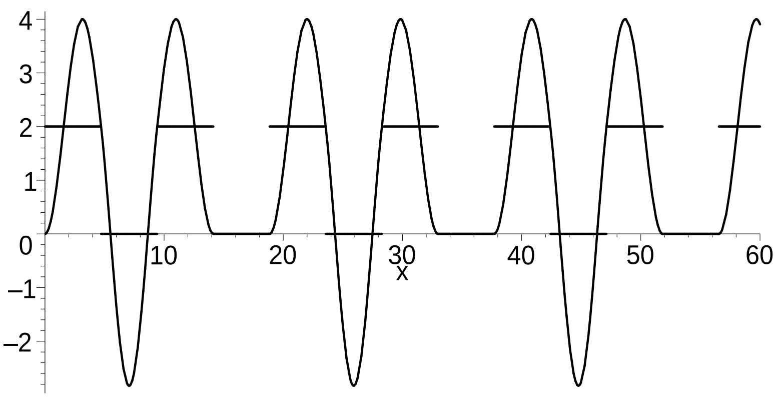


Figure 4.25: The graph of equation 9

solution is periodic with period $4c = 6\pi$. Notice that there is an interval where the motion of the mass is stopped. This occurs in the interval $[3c, 4c)$. The constant force applied on the interval $[2c, 3c)$ gently stops the motion of the mass by the time $t = 3c$. Since the force is 0 on $[3c, 4c)$ there is no movement. At $t = 4c$ the force is reapplied and the process thereafter repeats itself. This phenomenon occurs in all cases where the solution y is periodic. (cf. Exercise **??**)

In Section 3.7 we observed that when the natural frequency of the spring is close to but not equal to the frequency of the forcing function, $\cos(\omega t)$, then one observes vibrations that exhibit a beat. This phenomenon likewise occurs for the square wave forcing function. Let $r = 2$, $c = \frac{9\pi}{8}$, and $\beta = 1$. Recall that **frequency** is merely the reciprocal of the period so when these frequencies are close so are their periods. The natural period of the spring is $\frac{2\pi}{\beta} = 2\pi$ while the period of the forcing function is $2c = \frac{9\pi}{4}$: their periods are close and likewise their frequencies. Figure 4.26 gives a graph of y in this case. Again it is evident that the motion of the mass stops on the last

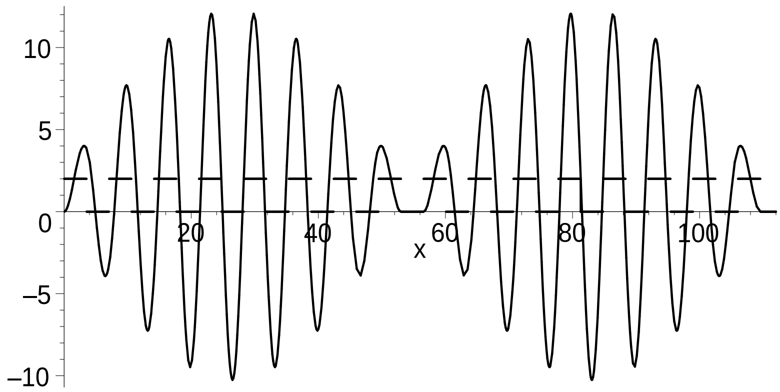


Figure 4.26: The graph of equation 9: the beats are evident here.

subinterval before the end of its period. More interesting is the fact that y oscillates with an amplitude that varies with time and produces 'beats'.

βc is an odd multiple of π

We now return to equation (6) in the case βc is an odd multiple of π. Things reduce substantially because $\cos(\beta t - N\beta c) = (-1)^N \cos(\beta t)$ and we get

$$\begin{aligned} f_2(t) &= \frac{r}{\beta^2} \sum_{N=0}^{\infty} \sum_{n=0}^{N} \cos(\beta t) \chi_{[Nc,(N+1)c)} \\ &= \frac{r}{\beta^2} \sum_{N=0}^{\infty} (N+1) \chi_{[Nc,(N+1)c)} \cos(\beta t) \\ &= \frac{r}{\beta^2} ([t/c]_1 + 1) \cos(\beta t). \end{aligned}$$

The solution now is

$$\begin{aligned} y(t) &= f_1(t) - f_2(t) \\ &= \frac{r}{\beta^2}\left(\mathrm{sw}_c(t) - [t/c]_1 \cos(\beta t) - \cos(\beta t)\right). \end{aligned} \tag{10}$$

Figure 4.27 gives the graph of this in the case where $r = 2$, $\beta = \pi$ and $c = 1$. Resonance

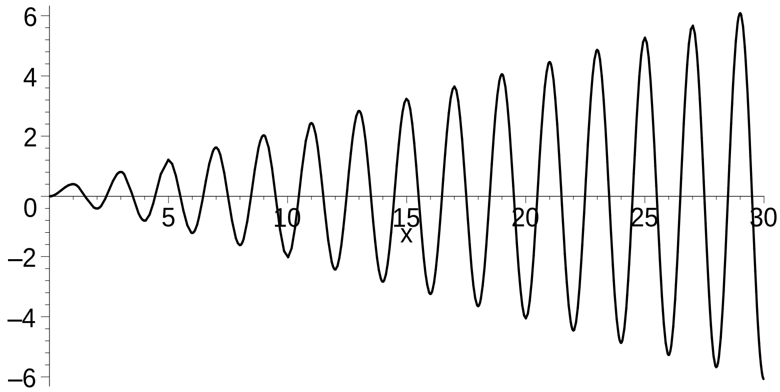


Figure 4.27: The graph of equation 10: resonance is evident here.

is clearly evident. Of course, this is an idealized situation; the spring would eventually fail.

◀

Undamped Motion with period impulses

Example 4.8.4. A mass-spring system with no damping force is acted upon at rest by an impulse force of r units at all multiples of c units of time starting at $t = 0$. (Imagine a hammer exerting blows to the mass at regular intervals.) Describe the motion of the mass.

▶ **Solution.** The differential equation that describes this system is given by

$$y'' + \beta^2 y = r \sum_{n=0}^{\infty} \delta_{nc} \quad y(0) = 0,\ y'(0) = 0,$$

where, again, β^2 is the spring constant. The Laplace transform gives

$$Y(s) = \frac{r}{\beta}\sum_{n=0}^{\infty} e^{-ncs}\frac{\beta}{s^2+\beta^2}.$$

By Theorem 4.7.6

$$\begin{aligned} y(t) &= \frac{r}{\beta}\sum_{n=0}^{\infty} \sin\beta(t-nc)h_{nc} \\ &= \frac{r}{\beta}\sum_{N=0}^{\infty}\sum_{n=0}^{N} \sin(\beta t - n\beta c)\chi_{[Nc,(N+1)c)} \qquad (11) \end{aligned}$$

Again we will consider two cases.

βc is not a multiple of 2π

Lemma 4.8.5. *Suppose v is not a multiple of 2π. Let $\alpha = \frac{\sin v}{1-\cos v}$. Then*

1. $\sum_{n=0}^{N} \sin(u+nv) = \frac{1}{2}\left(\sin u + \alpha\cos u + \sin(u+Nv) - \alpha\cos(u+Nv)\right).$
2. $\sum_{n=0}^{N} \cos(u+nv) = \frac{1}{2}\left(\cos u - \alpha\sin u + \cos(u+Nv) + \alpha\sin(u+Nv)\right).$

Let $u = \beta t$ and $v = -\beta c$. By the first part of Lemma 4.8.5 we get

$$\begin{aligned} y(t) &= \frac{r}{2\beta}\sum_{N=0}^{\infty}\left(\sin\beta t + \alpha\cos\beta t + \sin\beta(t-Nc) - \alpha\cos\beta(t-Nc)\right)\chi_{[Nc,(N+1)c)} \\ &= \frac{r}{2\beta}\left(\sin\beta t + \alpha\cos\beta t + \sin\beta < t >_c - \alpha\cos\beta < t >_c\right), \qquad (12) \end{aligned}$$

where $\alpha = \frac{-\sin\beta c}{1-\cos\beta c}$. Lemma 4.8.3 implies that the solution will be periodic when $\frac{c}{2\pi/\beta} = \frac{\beta c}{2\pi}$ is rational. Consider the following example. Let $r = 2$, $\beta = 1$ and $c = \frac{3\pi}{2}$. The graph of the solution, Equation (12), in this case is given in Figure 4.28. The period is $6\pi = 4c$. Observe that on the interval $[3c, 4c)$ the motion of the mass is completely stopped. At $t = 3c$ the hammer strikes and imparts a velocity that stops the mass dead in its track. At $t = 4c$ the process repeats itself. As in the previous example this phenomenon occurs in all cases where the solution y is periodic, i.e. when $\frac{c}{2\pi/(\beta)} = \frac{\beta c}{2\pi}$ is rational.

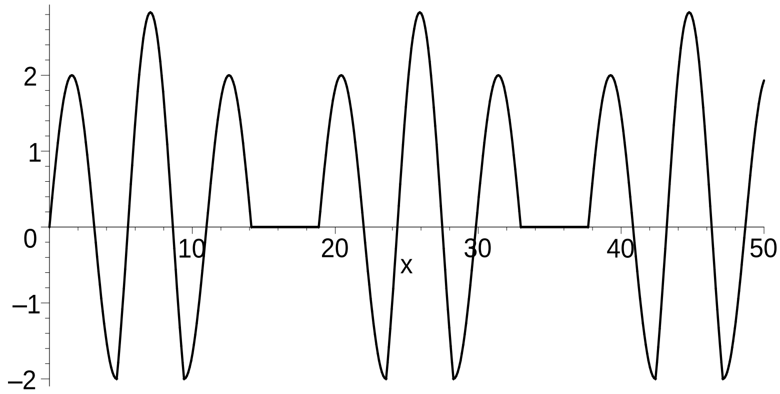


Figure 4.28: The graph of equation 12

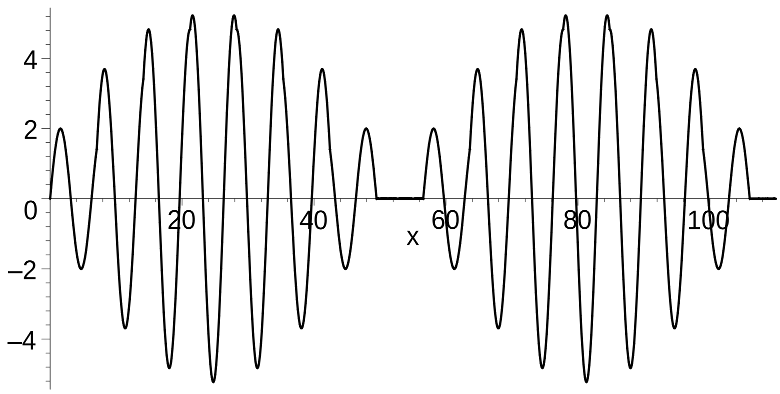


Figure 4.29: A solution that demonstrates beats.

When the period of the forcing function is close to that of the natural period of the spring the beats in the solution can again be seen. For example, Figure 4.29 shows the graph when $c = \frac{9}{8}(2\pi)$, $\beta = 1$, and $r = 2$.

βc is a multiple of 2π

In this case Equation (11) simplifies to

$$y(t) \quad = \quad \frac{r}{\beta}\left(\sin\beta t + [t/c]_1 \sin\beta t\right). \tag{13}$$

Figure 4.30 gives a graph of the solution when $c = 2\pi$, $\beta = 1$, and $r = 2$. In this case resonance occurs. ◀

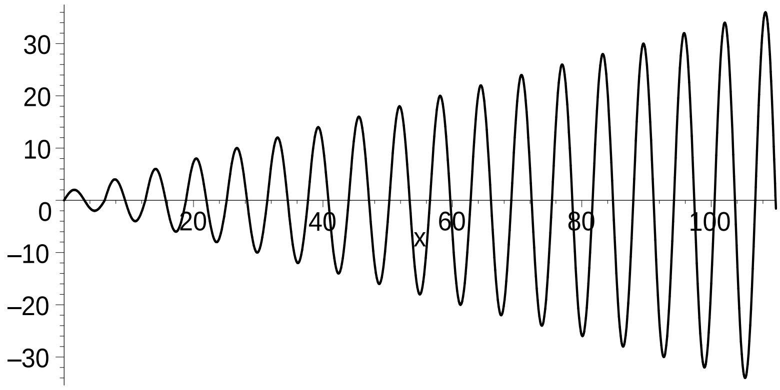


Figure 4.30: A solution with resonance.

4.9 Convolution

In this section we extend to the Heaviside class the definition of the convolution that we introduced in Section 2.4. The importance of the convolution is that it provides a closed formula for the inverse Laplace transform of a product of two functions. This is the essence of the convolution theorem which we give here. We will then consider further extensions to the delta functions δ_c and explore some very pleasant properties.

Given two functions f and g in $\mathcal{H}$ the function

$$u \mapsto f(u)g(t-u)$$

is continuous except for perhaps finitely many points on each interval of the form $[0,t]$. Therefore the integral

$$\int_0^t f(u)g(t-u)\,du$$

exists for each $t>0$. The **convolution** of f and g is given by

$$f * g(t) = \int_0^t f(u)g(t-u)\,du.$$

We will not make the argument but it can be shown that $f * g$ is in fact continuous. Since there are numbers K, L, a, and b such that

$$|f(t)| \leq Ke^{at} \quad \text{and} \quad |g(t)| \leq Le^{bt}$$

it follows that

$$\begin{aligned} |f * g(t)| &\leq \int_0^t |f(u)|\,|g(t-u)|\,du \\ &\leq KL\int_0^t e^{au}e^{b(t-u)}\,du \\ &= KLe^{bt}\int_0^t e^{(a-b)u}\,du \\ &= KL\begin{cases} te^{bt} & \text{if } a=b \\ \frac{e^{at}-e^{bt}}{a-b} & \text{if } a\neq b \end{cases}. \end{aligned}$$

This shows that $f * g$ is of exponential type and therefore is back in $\mathcal{H}$.

The linearity properties we listed in Section 2.4 extend to $\mathcal{H}$. We restate them here: Suppose f, g, and h are in $\mathcal{H}$. Then

1. $f * g \in \mathcal{H}$
2. $f * g = g * f$
3. $(f * g) * h = f * (g * h)$
4. $f * (g + h) = f * g + f * h$
5. $f * 0 = 0$.

The sliding window and an example

Let's now break the convolution up into its constituents to get a better idea of what it does. The function $u \mapsto g(-u)$ has a graph that is folded or flipped across the y-axis. The function $u \mapsto g(t-u)$ shifts the flip by $t \geq 0$. The convolution measures the amount of overlap between f and the flip and shift of g by positive values t. One can think of $g(t-u)$ as a horizontally sliding window by which f is examined and measured.

Example 4.9.1. Let $f(t) = t\chi_{[0,1)}(t)$ and $g(t) = \chi_{[1,2)}(t)$. Find the convolution $f * g$.

▶ **Solution.** The flip of g is $g(-u) = \chi_{[-2,-1)}(u)$ while the flip and shift of g is $g(t-u) = \chi_{[t-2,t-1)}(u)$. See Figure 4.31.

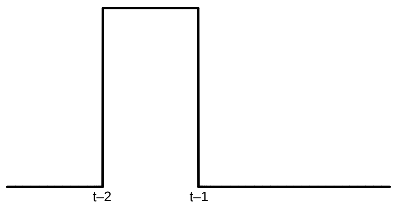


Figure 4.31: The flip and shift of $g = \chi_{[1,2)}$.

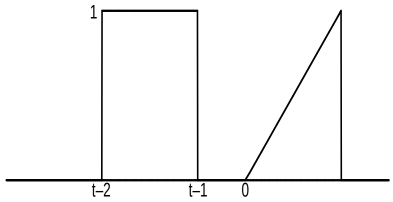


Figure 4.32: The window $g(t-u)$ and $f(u)$ have no overlap: $0 \leq t < 1$

If $t < 1$ then there is no overlap of the window $u \mapsto g(t-u)$ with f, i.e. $u \mapsto f(u)g(t-u) = 0$ and hence $f * g(t) = 0$. See Figure 4.32. Now suppose $1 \leq t < 2$. Then there is overlap between the window and f as seen in Figure 4.33.

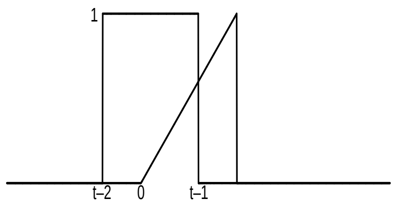


Figure 4.33: The window $g(t-u)$ and $f(u)$ overlap: $1 \leq t < 2$.

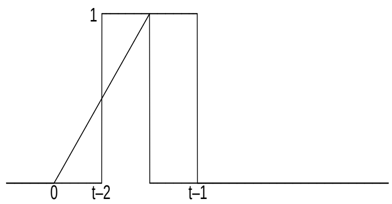


Figure 4.34: The window $g(t-u)$ and f continue to overlap: $2 \leq t < 3$.

The product of $f(u)$ and $g(t-u)$ is the function $u \mapsto u$, $0 \leq u < t-1$ and hence $f * g(t) = \frac{(t-1)^2}{2}$. Now if $2 \leq t < 3$ there is still overlap between the window and f as seen in Figure 4.34. The product of $f(u)$ and $g(t-u)$ is $u \mapsto u$, $t-2 \leq u < 1$ and $f * g(t) = \frac{1-(t-2)^2}{2} = \frac{-(t-1)(t-3)}{2}$. Finally, when $3 \leq t < \infty$ the window shifts past f as illustrated in Figure 4.35. The product of $f(u)$ and $g(t-u)$ is 0 and $f * g(t) = 0$.

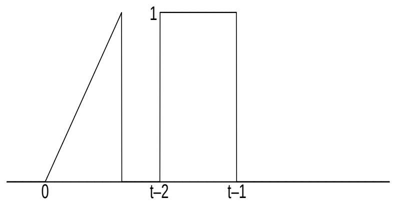


Figure 4.35: Again, there is no overlap: $3 \leq t < \infty$

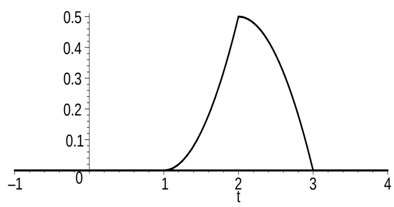


Figure 4.36: The convolution $f * g$.

We can now piece these function together to get

$$f * g(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ \frac{(t-1)^2}{2} & \text{if } 1 \le t < 2 \\ \frac{-(t-1)(t-3)}{2} & \text{if } 2 \le t < 3 \\ 0 & \text{if } 3 \le t < \infty \end{cases} = \frac{(t-1)^2}{2}\chi_{[1,2)} - \frac{(t-1)(t-3)}{2}\chi_{[2,3)}.$$

Its graph is given in Figure 4.36. Notice that the convolution is continuous; in this case it is not differentiable at $t = 2, 3$. ◀

Theorem 4.9.2 (The Convolution Theorem). *Suppose f and g are in $\mathcal{H}$ and F and G are their Laplace transforms, respectively. Then*

$$\mathcal{L}\left\{f * g\right\}(s) = F(s)G(s)$$

or, equivalently,

$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\}(t) = (f * g)(t).$$

Proof. For any $f \in \mathcal{H}$ we will define $f(t) = 0$ for $t < 0$. By Theorem 4.2.4

$$e^{-st}G(s) = \mathcal{L}\left\{g(u-t)h_t\right\}.$$

Therefore,

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-st} f(t)\, dt G(s) \\ &= \int_0^\infty e^{-st} G(s) f(t)\, dt \\ &= \int_0^\infty \mathcal{L}\left\{g(u-t)h_t(u)\right\}(s) f(t)\, dt \\ &= \int_0^\infty \int_0^\infty e^{-su} g(u-t)h(u-t)f(t)\, du\, dt \qquad (1) \end{aligned}$$

A theorem in calculus [1] tells us that we can switch the order of integration in (1)

[1] c.f. Vector Calculus, Linear Algebra, and Differential Forms, J.H. Hubbard and B.B Hubbard, page 444

when f and g are in $\mathcal{H}$. Thus we obtain

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^\infty e^{-su} g(u-t)h(u-t)f(t)\,dt\,du \\ &= \int_0^\infty \int_0^t e^{-su} g(u-t)f(t)\,dt\,du \\ &= \int_0^\infty e^{-su}(f*g)(u)\,du \\ &= \mathcal{L}\{f*g\}(s) \end{aligned}$$

□

There are a variety of uses for the convolution theorem. For one it is sometimes a convenient way to compute the convolution of two functions f and g; namely $(f*g)(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$.

Example 4.9.3. Compute the convolution of the functions given in 4.9.1:

$$f(t) = t\chi_{[0,1)} \quad \text{and} \quad g(t) = \chi_{[1,2)}.$$

In the following example, which is a reworking of Example 4.9.1, instead of keeping track of the sliding window $g(t-u)$ the convolution theorem turns the problem into one that is primarily algebraic.

▶ **Solution.** The Laplace transforms of f and g are, respectively,

$$F(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) \quad \text{and} \quad G(s) = \frac{e^{-s} - e^{-2s}}{s}.$$

The product simplifies to

$$F(s)G(s) = \frac{1}{s^3}e^{-s} - \left(\frac{2}{s^3} + \frac{1}{s^2}\right)e^{-2s} + \left(\frac{1}{s^3} + \frac{1}{s^2}\right)e^{-3s}.$$

Its inverse Laplace transform is

$$\begin{aligned} (f*g)(t) &= \mathcal{L}^{-1}\{F(s)G(s)\}(t) \\ &= \frac{(t-1)^2}{2}h_1(t) - ((t-2)(t-1))h_2(t) + \frac{(t-3)(t-1)}{2}h_3(t) \\ &= \frac{(t-1)^2}{2}\chi_{[1,2)}(t) - \frac{(t-1)(t-3)}{2}\chi_{[2,3)}(t) \end{aligned}$$

◀

Convolution and the Dirac Delta Function

We would like to extend the definition of convolution to include the delta functions δ_c, $c \geq 0$. Recall that we formally defined the delta function by

$$\delta_c(t) = \lim_{\epsilon \to 0} d_{c,\epsilon}(t),$$

where $d_{c,\epsilon} = \frac{1}{\epsilon}\chi_{[c,c+\epsilon)}$. In like manner, for $f \in \mathcal{H}$, we define

$$f * \delta_c(t) = \lim_{\epsilon \to 0} f * d_{c,\epsilon}(t).$$

Theorem 4.9.4. *For $f \in \mathcal{H}$*

$$f * \delta_c(t) = f(t-c)h_c,$$

where the equality is understood to mean essentially equal.

Proof. Let $f \in \mathcal{H}$. Then

$$\begin{aligned} f * d_{c,\epsilon}(t) &= \int_0^t f(u) d_{c,\epsilon}(t-u)\,dt \\ &= \frac{1}{\epsilon}\int_0^t f(u)\chi_{[c,c+\epsilon)}(t-u)\,du \\ &= \frac{1}{\epsilon}\int_0^t f(u)\chi_{[t-c-\epsilon,t-c)}(u)\,du \end{aligned}$$

Now suppose $t < c$. Then $\chi_{[t-c-\epsilon,t-c)}(u) = 0$, for all $u \in [0,t)$. Thus $f * d_{c,\epsilon} = 0$. On the other hand if $t > c$ then for ϵ small enough we have

$$f * d_{c,\epsilon}(t) = \frac{1}{\epsilon}\int_{t-c-\epsilon}^{t-c} f(u)\,du.$$

Let t be such that $t - c$ is a point of continuity of f. Then by the Fundamental Theorem of Calculus

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\int_{t-c-\epsilon}^{t-c} f(u)\,du = f(t-c).$$

Since f has only finitely many removable discontinuities on any finite interval it follows that $f * \delta_c$ is essentially equal to $f(t-c)h_c$. □

The special case $c = 0$ produces the following pleasant corollary.

Corollary 4.9.5. *For $f \in \mathcal{H}$ we have*

$$f * \delta_0 = f.$$

This corollary tells us that this extension to the Dirac delta function gives an identity under the convolution product. We thus have a correspondence between the multiplicative identities in domain and transform space under the Laplace transform since $\mathcal{L}\{\delta_0\} = 1$.

The Impulse Response Function

Let $f \in \mathcal{H}$. Let us return to our basic second order differential equation

$$ay'' + by' + cy = f(t), \quad y(0) = y_0 \text{ and } y'(0) = y_1. \tag{2}$$

By organizing terms in its Laplace transform in the right manner we can express the solution in terms of convolution of a special function called the impulse response function and f. To explain the main idea let's begin by considering the following special case

$$ay'' + by' + cy = 0 \quad y(0) = 0 \text{ and } y'(0) = 1.$$

This corresponds to a system in initial position but with a unit velocity. Our discussion in Section 4.6 shows that this is exactly the same thing as solving

$$ay'' + by' + cy = \delta_0, \quad y(0) = 0 \text{ and } y'(0) = 0$$

the same system at rest but with unit impulse at $t = 0$. The Laplace transform of either equation above leads to

$$Y(s) = \frac{1}{as^2 + bs + c}.$$

The inverse Laplace transform is the solution and will be denoted by $\zeta(t)$; it is called the **impulse response function**.

The Laplace transform of Equation 2 leads to

$$Y(s) = \frac{(as + b)y_0 + y_1}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}.$$

Let

$$H(s) = \frac{(as + b)y_0 + y_1}{as^2 + bs + c} \quad \text{and} \quad G(s) = \frac{F(s)}{as^2 + bs + c}.$$

Then $Y(s) = H(s) + G(s)$. The inverse Laplace transform of H corresponds to the solution to Equation 2 when $f = 0$. It is the homogeneous solution. On the other hand, G can be written as a product

$$G(s) = F(s)\left(\frac{1}{as^2 + bs + c}\right)$$

and its inverses Laplace transform $g(t)$ is

$$g(t) = f * \zeta(t),$$

by the convolution theorem.

We summarize this discussion in the following theorem:

Theorem 4.9.6. *Let $f \in \mathcal{H}$. The solution to Equation 2 can be expressed as*

$$h(t) + f * \zeta(t),$$

where h is the homogenous solution to Equation 2 and ζ is the impulse response function.

Example 4.9.7. Solve the following differential equation:

$$y'' + 4y = \chi_{[0,1)} \quad y(0) = 0 \text{ and } y'(0) = 0.$$

▶ **Solution.** The homogeneous solution to

$$y'' + 4y = 0 \quad y(0) = 0 \text{ and } y'(0) = 0$$

is the trivial solution $h = 0$. The impulse response function is

$$\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t.$$

By Theorem 4.9.6 the solution is

$$\begin{aligned} y(t) &= \zeta * \chi_{[0,1)} \\ &= \int_0^t \frac{1}{2}\sin(2u)\chi_{[0,1)}(t-u)\,du \\ &= \frac{1}{2}\int_0^t \sin(2u)\chi[t-1,t)(u)\,du \\ &= \frac{1}{2}\begin{cases} \int_0^t \sin 2u\,du & \text{if } 0 \le t < 1 \\ \int_{t-1}^t \sin 2u\,du & \text{if } 1 \le u < \infty \end{cases} \\ &= \frac{1}{4}\begin{cases} 1 - \cos 2t & \text{if } 0 \le t < 1 \\ \cos 2(t-1) - \cos 2t & \text{if } 1 \le t < \infty \end{cases} \end{aligned}$$

◀

Chapter 5

MATRICES

Most students by now have been exposed to the language of matrices. They arise naturally in many subject areas but mainly in the context of solving a simultaneous system of linear equations. In this chapter we will give a review of matrices, systems of linear equations, inverses, and determinants. The next chapter will apply what is learned here to linear systems of differential equations.

5.1 Matrix Operations

A **matrix** is a rectangular array of entities and is generally written in the following way:

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}.$$

We let $\mathcal{R}$ denote the set of entities that will be in use at any particular time. Each x_{ij} is in $\mathcal{R}$ and in this text $\mathcal{R}$ can be one of the following sets:

$\mathbb{R}$ or $\mathbb{C}$	The scalars
$\mathbb{R}[t]$ or $\mathbb{C}[t]$	Polynomials with real or complex entries
$\mathbb{R}(s)$ or $\mathbb{C}(s)$	The real or complex rational functions
$C^n(I,\mathbb{R})$ or $C^n(I,\mathbb{C})$	Real or complex valued functions with n continuous derivatives

Notice that addition and scalar multiplication is defined on $\mathcal{R}$. Below we will extend these operations to matrices. (In Chapter 6 we will see an instance where $\mathcal{R}$ will even be matrices themselves; thus matrices of matrices. But we will avoid that for now.)

The following are examples of matrices.

Example 5.1.1.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 9 \end{bmatrix} \quad C = \begin{bmatrix} i & 2-i \\ 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} t^2 e^{2t} \\ t^3 \cos t \end{bmatrix} \quad E = \begin{bmatrix} \frac{s}{s^2-1} & \frac{1}{s^2-1} \\ \frac{-1}{s^2-1} & \frac{s+2}{s^2-1} \end{bmatrix}$$

It is a common practice to use capital letters, like A, B, C, D, and E, to denote matrices. The **size** of a matrix is determined by the number of rows m and the number of columns n and written $m \times n$. In Example 5.1.1 A is a 2×3 matrix, B is a 1×3 matrix, C and E are 2×2 matrices, and D is a 2×1 matrix. A matrix is **square** if the number of rows is the same as the number of columns. Thus, C and E are square matrices. An entry in a matrix is determined by its position. If X is a matrix the (i, j) **entry** is the entry that appears in the i^{th} row and j^{th} column. We denote it in two ways: $\text{ent}_{ij}(X)$ or more simply X_{ij}. Thus, in Example 5.1.1, $A_{1\,3} = 3$, $B_{1\,2} = -1$, and $C_{2\,2} = 0$. We say that two matrices X and Y are **equal** if the corresponding entries are equal, i.e. $X_{i\,j} = Y_{i\,j}$, for all indices i and j. Necessarily X and Y must be the same size. The **main diagonal** of a square $n \times n$ matrix X is the vector formed from the entries $X_{i\,i}$, for $i = 1, \ldots, n$. The main diagonal of C is $(i, 0)$ and the main diagonal of E is $(\frac{s}{s^2-1}, \frac{s+2}{s^2-1})$. In this book all scalars are either real or complex. A matrix is said to be a **real matrix** if each entry is real and a **complex matrix** if each entry is complex. Since every real number is also complex, every real matrix is also a complex matrix. Thus A and B are real (and complex) matrices while C is a complex matrix.

Even though a matrix is a structured array of entities in $\mathcal{R}$ it should be viewed as a single object just as a word is a single object though made up of many letters. We let $M_{m,n}(\mathcal{R})$ denote the set of all $m \times n$ matrices with entries in $\mathcal{R}$. If the focus is on matrices of a certain size and not the entries we will sometimes just write $M_{m,n}$.

The following definitions highlights various kinds of matrices that commonly arise.

1. A **diagonal** matrix D is a square matrix in which all entries off the main diagonal are 0. We can say this in another way:

   $$D_{i\,j} = 0 \text{ if } i \neq j.$$

   Examples of diagonal matrices are:

   $$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{4t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{s} & 0 & 0 & 0 \\ 0 & \frac{2}{s-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{s-2} \end{bmatrix}.$$

It is convenient to write $\operatorname{diag}(d_1, \ldots, d_n)$ to represent the diagonal $n \times n$ matrix with $(d_1, \ldots, d_n)$ on the diagonal. Thus the diagonal matrices listed above are $\operatorname{diag}(1, 4)$, $\operatorname{diag}(e^t, e^{4t}, 1)$ and $\operatorname{diag}(\frac{1}{s}, \frac{2}{s-1}, 0, -\frac{1}{s-2})$, respectively.

2. The **zero** matrix $\mathbf{0}$ is the matrix with each entry 0. The size is usually determined by the context. If we need to be specific we will write $\mathbf{0}_{m,n}$ to mean the $m \times n$ zero matrix. Note that the square zero matrix, $\mathbf{0}_{n,n}$ is diagonal and is $\operatorname{diag}(0, \ldots, 0)$.

3. The **identity matrix**, I, is the square matrix with ones on the main diagonal and zeros elsewhere. The size is usually determined by the context, but if we want to be specific, we write I_n to denote the $n \times n$ identity matrix. The 2×2 and the 3×3 identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. We say a square matrix is **upper triangular** if each entry below the main diagonal is zero. We say a square matrix is **lower triangular** if each entry above the main diagonal is zero.

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{bmatrix} \quad \text{are upper triangular}$$

and

$$\begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -7 \end{bmatrix} \quad \text{are lower triangular.}$$

5. Suppose A is an $m \times n$ matrix. The **transpose** of A, denoted A^t, is the $n \times m$ matrix obtained by turning the rows of A into columns. In terms of the entries we have more explicitly,

$$(A^t)_{i\,j} = A_{j\,i}.$$

This expression reverses the indices of A and thus changes rows to columns and columns to rows. Simple examples are

$$\begin{bmatrix} 2 & 3 \\ 9 & 0 \\ -1 & 4 \end{bmatrix}^t = \begin{bmatrix} 2 & 9 & -1 \\ 3 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}^t = \begin{bmatrix} e^t & e^{-t} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{s} & \frac{2}{s^3} \\ \frac{2}{s^2} & \frac{3}{s} \end{bmatrix}^t = \begin{bmatrix} \frac{1}{s} & \frac{2}{s^2} \\ \frac{2}{s^3} & \frac{3}{s} \end{bmatrix}.$$

Matrix Algebra

There are three matrix operations that make up the algebraic structure of matrices: addition, scalar multiplication, and matrix multiplication.

Addition

Suppose A and B are two matrices of the same size. We define **matrix addition**, $A+B$, entrywise by the following formula

$$(A+B)_{ij} = A_{ij} + B_{ij}.$$

Thus if

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 5 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -1 & 0 \\ -3 & 8 & 1 \end{bmatrix}$$

then

$$A+B = \begin{bmatrix} 1+4 & -2-1 & 0+0 \\ 4-3 & 5+8 & -3+1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 0 \\ 1 & 13 & -2 \end{bmatrix}.$$

Corresponding entries are added. Addition preserves the size of matrices. We can symbolize this in the following way: $+ : M_{m,n}(\mathcal{R}) \times M_{m,n}(\mathcal{R}) \to M_{m,n}(\mathcal{R})$. Addition satisfies the following properties:

Proposition 5.1.2. *Suppose A, B, and C are $m \times n$ matrices. Then*

$$\begin{aligned} &A+B = B+A && \text{(commutative)} \\ &(A+B)+C = A+(B+C) && \text{(associative)} \\ &A+\mathbf{0} = A && \text{(additive identity)} \\ &A+(-A) = \mathbf{0} && \text{(additive inverse)} \end{aligned}$$

Scalar Multiplication

Suppose A is an matrix and $c \in \mathcal{R}$. We define **scalar multiplication**, $c \cdot A$, (but usually we will just write cA), entrywise by the following formula

$$(cA)_{ij} = cA_{ij}.$$

Thus if

$$c = -2 \text{ and } A = \begin{bmatrix} 1 & 9 \\ -3 & 0 \\ 2 & 5 \end{bmatrix}$$

then

$$cA = \begin{bmatrix} -2 & -18 \\ 6 & 0 \\ -4 & -10 \end{bmatrix}.$$

Scalar multiplication preserves the size of matrices. Thus $\cdot : \mathcal{R} \times M_{m,n}(\mathcal{R}) \to M_{m,n}(\mathcal{R})$. In this context we will call $c \in \mathcal{R}$ a scalar. Scalar multiplication satisfies the following properties:

Proposition 5.1.3. *Suppose A and B are matrices whose sizes are such that each line below is defined. Suppose $c_1, c_2 \in \mathcal{R}$. Then*

$$\begin{aligned} &c_1(A+B) = c_1A + c_1B && \text{(distributive)} \\ &(c_1+c_2)A = c_1A + c_2A && \text{(distributive)} \\ &c_1(c_2A) = (c_1c_2)A && \text{(associative)} \\ &1A = A \\ &0A = \mathbf{0} \end{aligned}$$

Matrix Multiplication

Matrix multiplication is more complicated than addition and scalar multiplication. We will define it in two stages: first on row and column matrices and then on general matrices.

A **row matrix** or **row vector** is a matrix which has only one row. Thus row vectors are in $M_{1,n}$. Similarly, a **column matrix** or **column vector** is a matrix which has only one column. Thus column vectors are in $M_{m,1}$. We frequently will denote column and row vectors by lower case boldface letters like $\mathbf{v}$ or $\mathbf{x}$ instead of capital letters. It is unnecessary to use double subscripts to indicate the entries of a row or column matrix: if $\mathbf{v}$ is a row vector then we write $\mathbf{v}_i$ for the i^{th} entry instead of $\mathbf{v}_{1\,i}$. Similarly for column vectors. Suppose $\mathbf{v} \in M_{1,n}$ and $\mathbf{w} \in M_{n,1}$. We define the product $\mathbf{v} \cdot \mathbf{w}$ (or preferably $\mathbf{vw}$) to be the scalar given by

$$\mathbf{vw} = v_1w_1 + \cdots + v_nw_n.$$

Even though this formula looks like the scalar product or dot product that you likely have seen before, keep in mind that $\mathbf{v}$ is a row vector while $\mathbf{w}$ is a column vector. For example, if

$$\mathbf{v} = \begin{bmatrix} 1 & 3 & -2 & 0 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 9 \end{bmatrix}$$

then

$$\mathbf{vw} = 1 \cdot 1 + 3 \cdot 3 + (-2) \cdot 0 + 0 \cdot 9 = 10.$$

Now suppose that A is any matrix. It is often convenient to distinguish the rows of A in the following way: If $\mathrm{Row}_i(A)$ denotes the i^{th} row of A then

$$A = \begin{bmatrix} \mathrm{Row}_1(A) \\ \mathrm{Row}_2(A) \\ \vdots \\ \mathrm{Row}_m(A) \end{bmatrix}.$$

Clearly $\mathrm{Row}_i(A)$ is a row vector. In a similar way, if B is another matrix we can distinguish the columns of B: Let $\mathrm{Col}_j(B)$ denote the j^{th} column of B then

$$B = \begin{bmatrix} \mathrm{Col}_1(B) & \mathrm{Col}_2(B) & \cdots & \mathrm{Col}_p(B) \end{bmatrix}.$$

Each $\mathrm{Col}_j(B)$ is a column vector.

Now let $A \in M_{mn}$ and $B \in M_{np}$. We define the **matrix product** of A and B to be the $m \times p$ matrix given entrywise by $\mathrm{ent}_{i\,j}(AB) = \mathrm{Row}_i(A)\,\mathrm{Col}_j(B)$. In other words, the (i, j)-entry of the product of A and B is the i^{th} row of A times the j^{th} column of B. We thus have

$$AB = \begin{bmatrix} \mathrm{Row}_1(A)\,\mathrm{Col}_1(B) & \mathrm{Row}_1(A)\,\mathrm{Col}_2(B) & \cdots & \mathrm{Row}_1(A)\,\mathrm{Col}_p(B) \\ \mathrm{Row}_2(A)\,\mathrm{Col}_1(B) & \mathrm{Row}_2(A)\,\mathrm{Col}_2(B) & \cdots & \mathrm{Row}_2(A)\,\mathrm{Col}_p(B) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A)\,\mathrm{Col}_1(B) & \mathrm{Row}_m(A)\,\mathrm{Col}_2(B) & \cdots & \mathrm{Row}_m(A)\,\mathrm{Col}_p(B) \end{bmatrix}.$$

Notice that each entry of AB is given as a product of a row vector and a column vector. Thus it is necessary that the number of columns of A (the first matrix) match the number of rows of B (the second matrix). This common number is n. The resulting product is an $m \times p$ matrix. Symbolically, $\cdot : M_{m,n}(\mathcal{R}) \times M_{n,p}(\mathcal{R}) \to M_{m,p}(\mathcal{R})$. In terms of the entries of A and B we have

$$\mathrm{ent}_{i\,j}(AB) = \mathrm{Row}_i(A)\,\mathrm{Col}_j(B) = \sum_{k=1}^{n} \mathrm{ent}_{i\,k}(A)\mathrm{ent}_{k\,j}(B) = \sum_{k=1}^{n} A_{i,k}B_{k,j}.$$

Example 5.1.4.

1. If

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$$

then AB is defined because the number of columns of A is the number of rows of B. Further AB is a 3×2 matrix and

$$AB = \begin{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ \\ \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} & \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ \\ \begin{bmatrix} 4 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} & \begin{bmatrix} 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 4 & -7 \\ 4 & 8 \end{bmatrix}.$$

2. If $A = \begin{bmatrix} e^t & 2e^t \\ e^{2t} & 3e^{2t} \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ then

$$AB = \begin{bmatrix} e^t(-2) + 2e^t(1) \\ e^{2t}(-2) + 3e^{2t}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}.$$

Notice in the definition (and the example) that in a given column of AB the corresponding column of B appears as the second factor. Thus

$$\operatorname{Col}_j(AB) = A \operatorname{Col}_j(B). \tag{1}$$

Similarly, in each row of AB the corresponding row of A appears and we get

$$\operatorname{Row}_i(A)B = \operatorname{Row}_i(AB). \tag{2}$$

Notice too that even though the product AB is defined it is not necessarily true that BA is defined. This is the case in part 1 of the above example due to the fact that the number of columns of B (2) does not match the number of rows of A (3). Even when AB and BA are defined it is not necessarily true that they are equal. Consider the following example:

Example 5.1.5. Suppose

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ 12 & -3 \end{bmatrix}$$

yet

$$BA = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 4 & 5 \end{bmatrix}.$$

These products are not the same. This example show that matrix multiplication is **not** commutative. However, the other properties that we are used to in an algebra are valid. We summarize them in the following proposition.

Proposition 5.1.6. *Suppose A, B, and C are matrices whose sizes are such that each line below is defined. Suppose $c_1, c_2 \in \mathcal{R}$. Then*

$$\begin{aligned} A(BC) &= (AB)C && \text{(associatvie)} \\ A(c_1B) &= (c_1A)B = c_1(AB) && \text{(associative)} \\ (A+B)C &= AC + BC && \text{(distributive)} \\ A(B+C) &= AB + AC && \text{(distributive)} \\ IA &= AI = I && (I \text{ is a multiplicative identity}) \end{aligned}$$

We highlight two useful formulas that follow from these algebraic properties. If A is an $m \times n$ matrix then

$$A\mathbf{x} = x_1 \operatorname{Col}_1(A) + \cdots x_n \operatorname{Col}_n(A), \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{3}$$

and

$$\mathbf{y}A = y_1 \mathrm{Row}_1(A) + \cdots y_m \mathrm{Row}_m(A), \qquad \text{where } \mathbf{y} = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix}. \tag{4}$$

Henceforth, we will use these algebraic properties without explicit reference. The following result expresses the relationship between multiplication and transposition of matrices

Theorem 5.1.7. *Let A and B be matrices such that AB is defined. Then B^tA^t is defined and*

$$B^tA^t = (AB)^t.$$

Proof. The number of columns of B^t is the same as the number of rows of B while the number of rows of A^t is the number of columns of A. These numbers agree since AB is defined so B^tA^t is defined. If n denotes these common numbers then

$$(B^tA^t)_{i\,j} = \sum_{k=1}^{n} (B^t)_{i\,k}(A^t)_{k\,j} = \sum_{k=1}^{n} A_{j\,k}B_{k\,i} = (AB)_{j\,i} = ((AB)^t)_{i\,j}.$$

□

Exercises

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 2 \\ -3 & 4 \\ 1 & 1 \end{bmatrix}$. Compute the following matrices.

1. $B + C$, $B - C$, $2B - 3C$

2. AB, AC, BA, CA

3. $A(B + C)$, $AB + AC$, $(B + C)A$

4. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$. Find C so that $3A + C = 4B$.

Let $A = \begin{bmatrix} 3 & -1 \\ 0 & -2 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 1 & -3 \\ 0 & -1 & 4 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & 8 \\ 1 & 1 & 7 \end{bmatrix}$. Find the following products

5. AB

6. BC

7. CA

8. $B^t A^t$

9. ABC.

10. Let $A = \begin{bmatrix} 1 & 4 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$. Find AB and BA.

Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$. Verify the following facts:

11. $A^2 = 0$

12. $B^2 = I_2$

13. $C^2 = C$

Compute $AB - BA$ in each of the following cases.

14. $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

15. $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{bmatrix}$

16. Let $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$. Show that there are no numbers a and b so that $AB - BA = I$, where I is the 2×2 identity matrix.

17. Suppose that A and B are 2×2 matrices.
    (a) Show by example that it need not be true that $(A + B)^2 = A^2 + 2AB + B^2$.
    (b) Find conditions on A and B to insure that the equation in Part (a) is valid.

18. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, compute A^2 and A^3.

19. If $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, compute B^n for all n.

20. If $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, compute A^2, A^3, and more generally, A^n for all n.

21. Let $A = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a matrix with two rows v_1 and v_2. (The number of columns of A is not relevant for this problem) Describe the effect of multiplying A on the left by the following matrices:

$$(a) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \quad (d) \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \quad (e) \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$$

22. Let $E(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Show that $E(\theta_1 + \theta_2) = E(\theta_1)E(\theta_2)$.

23. Let $(\theta) = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$. Show that $F(\theta_1 + \theta_2) = F(\theta_1)F(\theta_2)$.

24. Let $D = \operatorname{diag}(d_1, \ldots, d_n)$ and $E = \operatorname{diag}(e_1, \ldots, e_2)$. Show that

$$DE = \operatorname{diag}(d_1e_1, \ldots, d_ne_n)$$

.

5.2 Systems of Linear Equations

Most students have learned various techniques for finding the solution of a system of linear equations. They usually include various forms of elimination and substitutions. In this section we will learn the Gauss-Jordan elimination method. It is essentially a highly organized method involving elimination and substitution that always leads to the solution set. This general method has become the standard for solving systems. At first reading it may seem to be a bit complicated because of its description for general systems. However, with a little practice on a few examples it is quite easy to master. We will as usual begin with our definitions and proceed with examples to illustrate the needed concepts. To make matters a bit cleaner we will stick to the case where $\mathcal{R} = \mathbb{R}$. Everything we do here will work for $\mathcal{R} = \mathbb{C}$, $\mathbb{R}(s)$, or $\mathbb{C}(s)$ as well. (A technical difficulty for general $\mathcal{R}$ is the lack of inverses.)

If $x_1, \ldots, x_n$ are variables then the equation

$$a_1x_1 + \cdots + a_nx_n = b$$

is called a **linear equation** in the unknowns $x_1, \ldots, x_n$. A **system of linear equations** is a set of m linear equations in the unknowns $x_1, \ldots, x_n$ and is written in the form

$$\begin{array}{ccccccccc} a_{1\,1}x_1 & + & a_{1\,2}x_2 & + & \cdots & + & a_{1\,n}x_n & = & b_1 \\ a_{2\,1}x_1 & + & a_{2\,2}x_2 & + & \cdots & + & a_{2\,n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m\,1}x_1 & + & a_{m\,2}x_2 & + & \cdots & + & a_{m\,n}x_n & = & b_m. \end{array} \qquad (1)$$

The entries $a_{i\,j}$ are in $\mathbb{R}$ and are called **coefficients**. Likewise, each b_j is in $\mathbb{R}$. A key observation is that Equation (1) can be rewritten in matrix form as:

$$A\mathbf{x} = \mathbf{b}, \qquad (2)$$

where

$$A = \begin{bmatrix} a_{1\,1} & a_{1\,2} & \cdots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \cdots & a_{2\,n} \\ \vdots & \vdots & & \vdots \\ a_{m\,1} & a_{m\,2} & \cdots & a_{m\,n}, \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

We call A the **coefficient matrix**, $\mathbf{x}$ the **variable matrix**, and $\mathbf{b}$ the **output** matrix. Any column vector $\mathbf{x}$ with entries in $\mathcal{R}$ that satisfies (1) (or (2)) is called a **solution**. If a system has a solution we say it is **consistent**; otherwise, it is **inconsistent**. The **solution set**, denoted by $\mathcal{S}_A^{\mathbf{b}}$, is the set of all solutions. The system (1) is said to be **homogeneous** if $\mathbf{b} = \mathbf{0}$, otherwise it is called **nonhomogeneous**. Another important matrix associated with (2) is the **augmented matrix**:

$$[A|\, b\,] = \left[\begin{array}{cccc|c} a_{1\,1} & a_{1\,2} & \cdots & a_{1\,n} & b_1 \\ a_{2\,1} & a_{2\,2} & \cdots & a_{2\,n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m\,1} & a_{m\,2} & \cdots & a_{m\,n} & b_m \end{array}\right],$$

where the vertical line only serves to separate A from $\mathbf{b}$.

Example 5.2.1. Write the coefficient, variable, output, and augmented matrices for the following system:

$$\begin{array}{rcrcrcl} -2x_1 & + & 3x_2 & - & x_3 & = & 4 \\ x_1 & - & 2x_2 & + & 4x_3 & = & 5. \end{array}$$

Determine whether the following vectors are solutions:

$$\text{(a) } \mathbf{x} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \qquad \text{(b) } \mathbf{x} = \begin{bmatrix} 7 \\ 7 \\ 3 \end{bmatrix} \qquad \text{(c) } \mathbf{x} = \begin{bmatrix} 10 \\ 7 \\ 1 \end{bmatrix} \qquad \text{(d) } \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

► **Solution.** The coefficient matrix is $A = \begin{bmatrix} -2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix}$, the variable matrix is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the output matrix is $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and the augmented matrix is $\left[\begin{array}{rrr|r} -2 & 3 & -1 & 4 \\ 1 & -2 & 4 & 5 \end{array}\right]$. The system is nonhomogeneous. Notice that

$$A \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{and } A \begin{bmatrix} 7 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \text{ while } A \begin{bmatrix} 10 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore (a) and (b) are solutions, (c) is not a solution and the matrix in (d) is not the right size and thus cannot be a solution. ◄

Remark 5.2.2. When only 2 or 3 variables are involved in an example we will frequently use the variables x, y, and z instead of the subscripted variables x_1, x_2, and x_3.

Linearity

It is convenient to think of $\mathbb{R}^n$ as the set of column vectors $M_{n,1}(\mathbb{R})$. If A is an $m \times n$ real matrix then for each column vector $\mathbf{x} \in \mathbb{R}^n$, the product, $A\mathbf{x}$, is a column vector in $\mathbb{R}^m$. Thus the matrix A induces a map which we also denote just by $A : \mathbb{R}^n \to \mathbb{R}^m$ given by matrix multiplication. It satisfies the following important property.

Proposition 5.2.3. *The map $A : \mathbb{R}^n \to \mathbb{R}^m$ is linear. In other words,*

1. $A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y})$
2. $A(c\mathbf{x}) = cA(\mathbf{x})$,

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Proof. This follows directly from Propositions 5.1.3 and 5.1.6. □

Linearity is an extremely important property for it allows us to describe the structure of the solution set to $A\mathbf{x} = \mathbf{b}$ in a particularly nice way. Recall that $\mathcal{S}_A^{\mathbf{b}}$ denotes the solution set to the equation $A\mathbf{x} = \mathbf{b}$.

Proposition 5.2.4. *With A as above we have two possibilities:*

1. $\mathcal{S}_A^{\mathbf{b}} = \emptyset$ *or*
2. *there is an* $\mathbf{x}_p \in \mathcal{S}_A^{\mathbf{b}}$ *and* $\mathcal{S}_A^{\mathbf{b}} = \mathbf{x}_p + \mathcal{S}_A^{\mathbf{0}}$.

In other words, when $\mathcal{S}_A^{\mathbf{b}}$ is not the empty set then each solution to $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x}_p + \mathbf{x}_h,$$

where $\mathbf{x}_p$ is a fixed particular solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{0}$.

Proof. Suppose $\mathbf{x}_p$ is a fixed particular solution and $\mathbf{x}_h \in \mathcal{S}_A^{\mathbf{0}}$. Then $A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$. This implies that each column vector of the form $\mathbf{x}_p + \mathbf{x}_h$ is in $\mathcal{S}_A^{\mathbf{b}}$. On the other hand, suppose $\mathbf{x}$ is in $\mathcal{S}_A^{\mathbf{b}}$. Then $A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = 0$. This means that $\mathbf{x} - \mathbf{x}_p$ is in $\mathcal{S}_A^{\mathbf{0}}$. Therefore $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, for some vector $\mathbf{x}_h \in \mathcal{S}_A^{\mathbf{0}}$. □

Remark 5.2.5. The system of equations $A\mathbf{x} = \mathbf{0}$ is called the **associated homogeneous system**. Case (1) is a legitimate possibility. For example, the simple equation $0x = 1$ has empty solution set. When the solution set is not empty it should be mentioned that the particular solution $\mathbf{x}_p$ is not necessarily unique. In Chapter 3 we saw a similar theorem for a second order differential equation $Ly = f$. That theorem provided a strategy for solving such differential equations: First we solved the homogeneous equation $Ly = 0$ and second found a particular solution (using variation of parameters or undetermined coefficients). For a linear system of equations the matter is much simpler; the Gauss-Jordan method will give the whole solution set at one time. We will see that it has the above form.

Homogenous Systems

The homogeneous case, $A\mathbf{x} = \mathbf{0}$, is of particular interest. Observe that $\mathbf{x} = \mathbf{0}$ is always a solution so $\mathcal{S}_A^{\mathbf{0}}$ is never the empty set, i.e. case (1) is not possible. But much more is true.

Proposition 5.2.6. *The solution set, $\mathcal{S}_A^{\mathbf{0}}$, to a homogeneous system is closed under addition and multiplication by scalars. In other words, if $\mathbf{x}$ and $\mathbf{y}$ are solutions to the homogeneous system and c is a scalar then $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$ are also solutions.*

Proof. Suppose $\mathbf{x}$ and $\mathbf{y}$ are in $\mathcal{S}_A^{\mathbf{0}}$. Then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. This shows that $\mathbf{x} + \mathbf{y}$ is in $\mathcal{S}_A^{\mathbf{0}}$. Now suppose $c \in \mathcal{R}$. Then $A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}$. Hence $c\mathbf{x} \in \mathcal{S}_A^{\mathbf{0}}$. This shows that $\mathcal{S}_A^{\mathbf{0}}$ is closed under addition and scalar multiplication. □

Corollary 5.2.7. *The solution set to a general system of linear equations, $A\mathbf{x} = \mathbf{b}$, is either*

1. *empty*
2. *unique*
3. *or infinite.*

Proof. The associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has solution set, $\mathcal{S}_A^{\mathbf{0}}$, that is either equal to the trivial set $\{\mathbf{0}\}$ or an infinite set. To see this suppose that $\mathbf{x}$ is a nonzero solution to $A\mathbf{x} = \mathbf{0}$ then by Proposition 5.2.6 all multiples, $c\mathbf{x}$, are in $\mathcal{S}_A^{\mathbf{0}}$ as well. Therefore, by Proposition 5.2.4, if there is a solution to $A\mathbf{x} = \mathbf{b}$ it is unique or there are infinitely many. ☐

The Elementary Equation and Row Operations

We say that two systems of equations are **equivalent** if their solution sets are the same. This definition implies that the variable matrix is the same for each system.

Example 5.2.8. Consider the following systems of equations:

$$\begin{array}{rcrcr} 2x & + & 3y & = & 5 \\ x & - & y & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} x & = & 1 \\ y & = & 1. \end{array}$$

The solution set to the second system is transparent. For the first system there are some simple operations that easily lead to the solution: First, switch the two equations around. Next, multiply the equation $x - y = 1$ by -2 and add the result to the first. We then obtain

$$\begin{array}{rcrcr} x & - & y & = & 0 \\ & & 5y & = & 5 \end{array}$$

Next, multiply the second equation by $\frac{1}{5}$ to get $y = 1$. Then add this equation to the first. We get $x = 1$ and $y = 1$. Thus they both have the same solution set, namely the single vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. They are thus equivalent. When used in the right way these kinds of operations can transform a complicated system into a simpler one. We formalize these operations in the following definition:

Suppose $A\mathbf{x} = \mathbf{b}$ is a given system of linear equations. The following three operations are called **elementary equation operations**.

1. Switch the order in which two equations are listed
2. Multiply an equation by a nonzero scalar
3. Add a multiple of one equation to another

Notice that each operation produces a new system of linear equations but leaves the size of the system unchanged. Furthermore we have the following proposition.

Proposition 5.2.9. *An elementary equation operation applied to a system of linear equations is an equivalent system of equations.*

Proof. This means that the system that arises from an elementary equation operation has the same solution set as the original. We leave the proof as an exercise. □

The main idea in solving a system of linear equations is to perform a finite sequence of elementary equation operations to transform a system into simpler system where the solution set is transparent. Proposition 5.2.9 implies that the solution set of the simpler system is the same as original system. Let's consider our example above.

Example 5.2.10. Use elementary equation operations to transform

$$\begin{array}{ccccc} 2x & + & 3y & = & 5 \\ x & - & y & = & 0 \end{array}$$

into

$$\begin{array}{ccc} x & = & 1 \\ y & = & 1. \end{array}$$

▶ **Solution.**

$$\begin{array}{ccccc} 2x & + & 3y & = & 5 \\ x & - & y & = & 0 \end{array}$$

Switch the order of the two equations

$$\begin{array}{ccccc} x & - & y & = & 0 \\ 2x & + & 3y & = & 5 \end{array}$$

Add -2 times the first equation to the second equation

$$\begin{array}{ccccc} x & - & y & = & 0 \\ & & 5y & = & 5 \end{array}$$

Multiply the second equation by $\frac{1}{5}$

$$\begin{array}{ccccc} x & - & y & = & 0 \\ & & y & = & 1 \end{array}$$

Add the second equation to the first

$$\begin{array}{ccccc} x & & & = & 1 \\ & & y & = & 1 \end{array}$$

◀

Each operation produces a new system equivalent to the first by Proposition (5.2.9). The end result is a system where the solution is transparent. Since $y = 1$ is apparent in the fourth system we could have stopped and used the method of **back substitution**, that is, substitute $y = 1$ into the first equation and solve for x. However, it is in accord with the Gauss-Jordan elimination method to continue as we did to eliminate the variable y in the first equation.

You will notice that the variables x and y play no prominent role here. They merely serve as placeholders for the coefficients, some of which change with each operation. We thus simplify the notation (and the amount of writing) by performing the elementary operations on just the augmented matrix. The elementary equation operations become the **elementary row operations** which act on the augmented matrix of the system.

The **elementary row operations** on a matrix are

1. Switch two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of one row to another.

The following notations for these operations will be useful.

1. p_{ij} - switch rows i and j.
2. $m_i(a)$ - multiply row i by $a \neq 0$.
3. $t_{ij}(a)$ - add to row j the value of a times row i.

The effect of p_{ij} on a matrix A is denoted by $p_{ij}(A)$. Similarly for the other elementary row operations.

The corresponding operations when applied to the augmented matrix for the system in example 5.2.10 becomes:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{p_{12}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 3 & 5 \end{array}\right] \xrightarrow{t_{12}(-2)} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{m_2(1/5)} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{t_{21}(1)} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

Above each arrow is the notation for the elementary row operation performed to produce the next augmented matrix. The sequence of elementary row operations chosen follows a certain strategy: Starting from left to right and top down one tries to isolate a 1 in a given column and produce 0's above and below it. This corresponds to isolating and eliminating variables.

Let's consider three illustrative examples. The sequence of elementary row operation we perform is in accord with the Gauss-Jordan method which we will discuss in detail later on in this section. For now verify each step. The end result will be an equivalent system for which the solution set will be transparent.

Example 5.2.11. Consider the following system of linear equations

$$\begin{array}{ccccccc} 2x & + & 3y & + & 4z & = & 9 \\ x & + & 2y & - & z & = & 2 \end{array}.$$

Find the solution set and write it in the form $\mathbf{x}_p + \mathcal{S}_A^{\mathbf{0}}$.

► **Solution.** We first will write the augmented matrix and perform a sequence of elementary row operations:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 9 \\ 1 & 2 & -1 & 2 \end{array}\right] \xrightarrow{p_{1\,2}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 3 & 4 & 9 \end{array}\right] \xrightarrow{t_{1\,2}(-2)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 6 & 5 \end{array}\right]$$

$$\xrightarrow{m_2(-1)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -6 & -5 \end{array}\right] \xrightarrow{t_{2\,1}(-2)} \left[\begin{array}{ccc|c} 1 & 0 & 11 & 12 \\ 0 & 1 & -6 & -5 \end{array}\right]$$

The last augmented matrix corresponds to the system

$$\begin{array}{ccccccc} x & + & & & 11z & = & 12 \\ & & y & - & 6z & = & -5. \end{array}$$

In the first equation we can solve for x in terms of z and in the second equation we can solve for y in terms of z. We refer to z as a **free variable** and let $z = \alpha$ be a parameter in $\mathbb{R}$. Then we obtain

$$\begin{array}{ccc} x & = & 12 - 11\alpha \\ y & = & -5 + 6\alpha \\ z & = & \alpha \end{array}$$

In vector form we write

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 - 11\alpha \\ -5 + 6\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 12 \\ -5 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -11 \\ 6 \\ 1 \end{bmatrix}.$$

The vector, $\mathbf{x}_p = \begin{bmatrix} 12 \\ -5 \\ 0 \end{bmatrix}$ is a particular solution (corresponding to $\alpha = 0$) while the vector $\mathbf{x}_h = \begin{bmatrix} -11 \\ 6 \\ 1 \end{bmatrix}$ generates the homogeneous solutions as α varies over $\mathbb{R}$. We have thus written the solution in the form $\mathbf{x}_p + \mathcal{S}_A^0$. ◄

Example 5.2.12. Find the solution set for the system

$$\begin{array}{rcrcrcr} 3x & + & 2y & + & z & = & 4 \\ 2x & + & 2y & + & z & = & 3 \\ x & + & y & + & z & = & 0. \end{array}$$

▶ **Solution.** Again we start with the augmented matrix and apply elementary row operations. Occasionally we will apply more than one operation at a time. When this is so we stack the operations above the arrow with the topmost operation performed first followed in order by the ones below it.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{p_{13}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 3 \\ 3 & 2 & 1 & 4 \end{array}\right] \xrightarrow[t_{13}(-3)]{t_{12}(-2)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & -1 & -2 & 4 \end{array}\right]$$

$$\xrightarrow{p_{23}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & -1 & 3 \end{array}\right] \xrightarrow[m_3(-1)]{m_2(-1)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow[t_{31}(-1)]{t_{32}(-2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{t_{21}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The last augmented matrix corresponds to the system

$$\begin{array}{rcr} x & = & 1 \\ y & = & 2 \\ z & = & -3. \end{array}$$

The solution set is transparent: $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$. ◀

In this example we note that $S_A^{\mathbf{0}} = \{\mathbf{0}\}$ so that the solution set $S_A^{\mathbf{b}}$ consists of a single point: The system has a unique solution.

Example 5.2.13. Solve the following system of linear equations:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 4z & = & -2 \\ x & + & y & + & 3z & = & 1 \\ 2x & + & y & + & 5z & = & 2 \end{array}$$

▶ **Solution.** Again we begin with the augmented matrix and perform elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 1 & 1 & 3 & 1 \\ 2 & 1 & 5 & 2 \end{array}\right] \xrightarrow[t_{13}(-2)]{t_{12}(-1)} \left[\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 0 & -1 & -1 & 3 \\ 0 & -3 & -3 & 6 \end{array}\right] \xrightarrow{m_2(-1)} \left[\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & -3 & -3 & 6 \end{array}\right]$$

$$\xrightarrow{t_{23}(3)} \left[\begin{array}{ccc|c} 1 & 2 & 4 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & -3 \end{array}\right] \xrightarrow[t_{21}(-2)]{m_3(-1/3)} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[t_{32}(3)]{t_{31}(-6)} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The system that corresponds to the last augmented matrix is

$$\begin{array}{rcrcl} x & + & 2z & = & 0 \\ y & + & z & = & 0 \\ & & 0 & = & 1. \end{array}$$

The last equation, which is shorthand for $0x+0y+0z=1$, clearly has no solution. Thus the system has no solution. In this case we write $\mathcal{S}_A^{\mathbf{b}} = \emptyset$. ◀

Reduced Matrices

These last three examples typify what happens in general and illustrate the three possible outcomes discussed in Corollary 5.2.7: infinitely many solutions, a unique solution, or no solution at all. The most involved case is when the solution set has infinitely many solutions. In Example 5.2.11 a single parameter α was needed to generate the set of solutions. However, in general, there may be many parameters needed. We will always want to use the least number of parameters possible, without dependencies amongst them. In each of the three preceding examples it was transparent what the solution was by considering the system determined by the last listed augmented matrix. The last matrix was in a certain sense reduced as simple as possible.

We say that a matrix A is in **row echelon form (REF)** if the following three conditions are satisfied.

1. The nonzero rows lie above the zero rows.

2. The first nonzero entry in a non zero row is 1. (We call such a 1 a **leading one**.)

3. For any two adjacent nonzero rows the leading one of the upper row is to the left of the leading one of the lower row. (We say the leading ones are in echelon form.)

We say A is in **row reduced echelon form (RREF)** if it also satisfies

4 The entries above each leading one are zero.

Example 5.2.14. Determine which of the following matrices are row echelon form, row reduced echelon form, or neither. For the matrices in row echelon form determine the columns (**C**) of the leading ones. If a matrix is not in row reduced echelon form explain which conditions are violated.

$$(1)\ \begin{bmatrix} 1 & 0 & -3 & 11 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad (2)\ \begin{bmatrix} 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3)\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(4)\ \begin{bmatrix} 1 & 0 & 0 & 4 & 3 & 0 \\ 0 & 2 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5)\ \begin{bmatrix} 1 & 1 & 2 & 4 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

► **Solution.**

1. (REF): leading ones are in the first, third and fourth column. It is not reduced because there is a nonzero entry above the leading one in the third column.

2. (RREF): The leading ones are in the second and third column.

3. neither: The zero row is not at the bottom.

4. neither: The first non zero entry in the second row is not 1.

5. (REF): leading ones are in the first and fifth column. It is not reduced because there is a nonzero entry above the leading one in the fifth column.

6. neither: The leading ones are not in echelon form.

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The definitions we have given are for arbitrary matrices and not just matrices that come from a system of linear equations; i.e. the augmented matrix. Suppose though that a system $A\mathbf{x} = \mathbf{b}$ which has solutions is under consideration. If the augmented matrix $[A|\mathbf{b}]$ is transformed by elementary row operations to a matrix which is in row reduced echelon form the variables that correspond to the columns where the leading ones occur are called the **leading variables** or **dependent variables**. All of the other variables are called **free variables**. The free variables are sometimes replaced by parameters, like $\alpha, \beta, \ldots$. Each leading variable can be solved for in terms of the free variables alone. As the parameters vary the solution set is generated. The Gauss-Jordan elimination method which will be explained shortly will always transform an augmented matrix into a matrix that is in row reduced echelon form. This we did in Examples 5.2.11, 5.2.12, and 5.2.13. In Example 5.2.11 the augmented matrix was transformed to

$$\left[\begin{array}{ccc|c} 1 & 0 & 11 & 12 \\ 0 & 1 & -6 & -5 \end{array}\right].$$

The leading variables are x and y while there is only one free variable, z. Thus we obtained

$$\mathbf{x} = \begin{bmatrix} 12 - 11\alpha \\ -5 + 6\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 12 \\ -5 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -11 \\ 6 \\ 1 \end{bmatrix},$$

where z is replace by the parameter α. In example 5.2.12 the augmented matrix was transformed to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right].$$

In this case x, y, and z are leading variables; there are no free variables. The solution set is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

In Example 5.2.13 the augmented matrix was transformed to

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

In this case there are no solutions; the last row corresponds to the equation $0 = 1$. There are no leading variable nor free variables.

These examples illustrate the following proposition which explains Corollary 5.2.7 in terms of the augmented matrix in row reduced echelon form.

Proposition 5.2.15. *Suppose $A\mathbf{x} = \mathbf{b}$ is a system of linear equations and the augmented matrix $[A|\mathbf{b}]$ is transformed by elementary row operations to a matrix $[A'|\mathbf{b}']$ which is in row reduced echelon form.*

1. *If a row of the form $\begin{bmatrix} 0 & \dots & 0 & | & 1 \end{bmatrix}$ appears in $[A'|\mathbf{b}']$ then there are no solutions.*
2. *If there are no rows of the form $\begin{bmatrix} 0 & \dots & 0 & | & 1 \end{bmatrix}$ and no free variables associated with $[A'|\mathbf{b}']$ then there is a unique solution.*
3. *If there is one or more free variables associated with $[A'|\mathbf{b}']$ and no rows of the form $\begin{bmatrix} 0 & \dots & 0 & | & 1 \end{bmatrix}$ then there are infinitely many solution.*

Example 5.2.16. Suppose the following matrices are obtained by transforming the augmented matrix of a system of linear equations using elementary row operations. Identify the leading and free variables and write down the solution set. Assume the variables are $x_1, x_2, \dots$.

$$(1) \left[\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (2) \left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (3) \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$(4) \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad (5) \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad (6) \left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

► **Solution.** 1. The zero row provides no information and can be ignored. The variables are x_1, x_2, x_3, and $\mathbf{x}_4$. The leading ones occur in the first and fourth column. Therefore x_1 and x_4 are the leading variables. The free variables are x_2 and x_3. Let $\alpha = x_2$ and $\beta = x_3$. The first row implies the equation $x_1+x_2+4x_3 = 2$. We solve for x_1 and obtain $x_1 = 2-x_2-4x_3 = 2-\alpha-4\beta$. The second row implies the equation $x_4 = 3$. Thus

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2-\alpha-4\beta \\ \alpha \\ \beta \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

where α and β are arbitrary parameters in $\mathcal{R}$.

2. The leading ones are in the first and second column therefore x_1 and x_2 are the leading variables. The free variables of x_3 and x_4. Let $\alpha = x_3$ and $\beta = x_4$. The

first row implies $x_1 = 2 - 3\alpha - \beta$ and the second row implies $x_2 = 3 - \alpha + \beta$. The solution is

$$\mathbf{x} = \begin{bmatrix} 2-3\alpha-\beta \\ 3-\alpha+\beta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

where α, β are in $\mathcal{R}$.

3. x_1 is the leading variable. $\alpha = x_2$ and $\beta = x_3$ are free variables. The first row implies $x_1 = 1 - \alpha$. The solution is

$$\mathbf{x} = \begin{bmatrix} 1-\alpha \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where α and β are in $\mathcal{R}$.

4. The leading variables are x_1, x_2, and x_3. There are no free variables. The solution set is

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

5. The row $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ implies the solution set is empty.

6. The leading variables are x_2 and x_4. The free variables are $\alpha = x_1$ and $\beta = x_3$. The first row implies $x_2 = 2 - 2\beta$ and the second row implies $x_4 = 0$. The solution set is

$$\mathbf{x} = \begin{bmatrix} \alpha \\ 2-2\beta \\ \beta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix},$$

where α and β are in $\mathcal{R}$.

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The Gauss-Jordan Elimination Method

Now that you have seen several examples we present the Gauss-Jordan Elimination Method for any matrix. It is an algorithm to transform any matrix to row reduced echelon form using a finite number of elementary row operations. When applied to an augmented matrix of a system of linear equations the solution set can be readily discerned. It has other uses as well so our description will be for an arbitrary matrix.

Algorithm 5.2.17. The Gauss-Jordan Elimination Method Let A be a matrix. There is a finite sequence of elementary row operations that transform A to a matrix in row reduced echelon form. There are two stages of the process: (1) The first stage is called **Gaussian elimination** and transforms a given matrix to row echelon form and (2) The second stage is called **Gauss-Jordan elimination** and transforms a matrix in row echelon form to row reduced echelon form.

From A to REF: Gaussian elimination

1. Let $A_1 = A$. If $A_1 = \mathbf{0}$ then A is in row echelon form.
2. If $A_1 \neq 0$ then in the first nonzero column from the left, (say the j^{th} column) locate a nonzero entry in one of the rows: (say the i^{th} row with entry a.)
   (a) Multiply that row by the reciprocal of that nonzero entry. ($m_i(1/a)$)
   (b) Permute that row with the top row. ($p_{1\,i}$) There is now a 1 in the $(1, j)$ entry.
   (c) If b is a nonzero entry in the (i, j) position for $i \neq 1$, add $-b$ times the first row to the i^{th} row.($t_{1\,j}(-b)$) Do this for each row below the first.
   The transformed matrix will have the following form
$$\begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & & & \\ \vdots & \ddots & \vdots & & & A_2 & \\ 0 & \cdots & 0 & 0 & & & \end{bmatrix}.$$
   The *'s in the first row are unknown entries and A_2 is a matrix with fewer rows and columns than A_1.
3. If $A_2 = \mathbf{0}$ we are done. The above matrix in in row echelon form.
4. If $A_2 \neq \mathbf{0}$, apply step (2) to A_2. Since there are zeros to the left of A_2 and the only elementary row operations we apply effect the rows of A_2 (and not all of A) there will continue to be zeros to the left of A_2. The result will be a matrix of the form
$$\begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & * & * & \cdots & * \\ & & & 0 & 0 & \cdots & 0 & 1 & * & \cdots & * \\ \vdots & \ddots & \vdots & 0 & 0 & \cdots & 0 & 0 & & & \\ & & & \vdots & \vdots & & \vdots & \vdots & & A_3 & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & & & \end{bmatrix}.$$

5. If $A_3 = 0$, we are done. Otherwise continue repeating step (2) until a matrix $A_k = \mathbf{0}$ is obtained.

From REF to RREF: Gauss-Jordan Elimination

1. The leading ones now become apparent in the previous process. We begin with the rightmost leading one. Suppose it is in the k^{th} row and l^{th} column. If there is a nonzero entry (b say) above that leading one we add $-b$ times the k^{th} row to it. ($t_{k\,j}(-b)$.) We do this for each nonzero entry in the l^{th} column. The result is zeros above the rightmost leading one. (The entries to the left of a leading one are zeros. This process preserves that property.)

2. Now repeat the process described above to each leading one moving right to left. The result will be a matrix in row reduced echelon form.

Example 5.2.18. Use the Gauss-Jordan method to row reduce the following matrix to echelon form:

$$\begin{bmatrix} 2 & 3 & 8 & 0 & 4 \\ 3 & 4 & 11 & 1 & 8 \\ 1 & 2 & 5 & 1 & 6 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

► **Solution.** We will first write out the sequence of elementary row operations that transforms A to row reduced echelon form.

$$\begin{bmatrix} 2 & 3 & 8 & 0 & 4 \\ 3 & 4 & 11 & 1 & 8 \\ 1 & 2 & 5 & 1 & 6 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{p_{13}} \begin{bmatrix} 1 & 2 & 5 & 1 & 6 \\ 3 & 4 & 11 & 1 & 8 \\ 2 & 3 & 8 & 0 & 4 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow[t_{1\,4}(1)]{t_{1\,2}(-3)\; t_{1\,3}(-2)}$$

$$\begin{bmatrix} 1 & 2 & 5 & 1 & 6 \\ 0 & -2 & -4 & -2 & -10 \\ 0 & -1 & -2 & -2 & -8 \\ 0 & 2 & 4 & 1 & 7 \end{bmatrix} \xrightarrow{m_2(-1/2)} \begin{bmatrix} 1 & 2 & 5 & 1 & 6 \\ 0 & 1 & 2 & 1 & 5 \\ 0 & -1 & -2 & -2 & -8 \\ 0 & 2 & 4 & 1 & 7 \end{bmatrix} \xrightarrow[t_{2\,4}(-2)]{t_{2\,3}(1)}$$

$$\begin{bmatrix} 1 & 2 & 5 & 1 & 6 \\ 0 & 1 & 2 & 1 & 5 \\ 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -1 & -3 \end{bmatrix} \xrightarrow[t_{3\,4}(1)]{m_3(-1)} \begin{bmatrix} 1 & 2 & 5 & 1 & 6 \\ 0 & 1 & 2 & 1 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[t_{3\,1}(-1)]{t_{3\,2}(-1)}$$

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 3 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{t_{2\,1}(-2)} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the first step we observe that the first column is nonzero so it is possible to produce a 1 in the upper left hand corner. This is most easily accomplished by $p_{1,3}$. The next set of operations produces 0's below this leading one. We repeat this procedure on the submatrix to the right of the zeros's. We produce a one in the $2, 2$ position by $m_2(-\frac{1}{2})$ and the next set of operations produce zeros below this second leading one. Now notice that the third column below the second leading one is zero. There are no elementary row operations that can produce a leading one in the $(3, 3)$ position that involve just the third and fourth row. We move over to the fourth column and observe that the entries below the second leading one are not both zero. The elementary row operation $m_3(-1)$ produces a leading one in the $(3, 4)$ position and the subsequent operation produces a zero below it. At this point A has been transformed to row echelon form. Now starting at the rightmost leading one, the 1 in the $3, 4$ position, we use operations of the form $t_{3\,i}(a)$ to produce zeros above that leading one. This is applied to each column that contains a leading one. ◀

The student is encouraged to go carefully through Examples 5.2.11, 5.2.12, and 5.2.13. In each of those examples the Gauss-Jordan Elimination method was used to transform the augmented matrix to the matrix in row reduced echelon form.

Exercises

1. For each system of linear equations identify the coefficient matrix A, the variable matrix $\mathbf{x}$, the output matrix $\mathbf{b}$ and the augmented matrix $[A|\mathbf{b}]$.

$$(a)\quad \begin{array}{rcrcrcl} x & + & 4y & + & 3z & = & 2 \\ x & + & y & - & z & = & 4 \\ 2x & + & & & z & = & 1 \\ & & y & - & z & = & 6 \end{array} \qquad (b)\quad \begin{array}{rcrcrcrcl} 2x_1 & - & 3x_2 & + & 4x_3 & + & x_4 & = & 0 \\ 3x_1 & + & 8x_2 & - & 3x_3 & - & 6x_4 & = & 1 \end{array}$$

2. Suppose $A = \begin{bmatrix} 1 & 0 & -1 & 4 & 3 \\ 5 & 3 & -3 & -1 & -3 \\ 3 & -2 & 8 & 4 & -3 \\ -8 & 2 & 0 & 2 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -4 \end{bmatrix}$. Write out the system of linear equations that corresponds to $A\mathbf{x} = \mathbf{b}$.

In the following matrices identify those that are in row reduced echelon form. If a matrix is not in row reduced echelon form find a single elementary row operation that will transform it to row reduced echelon form and write the new matrix.

3. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 3 & 1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 4 & 1 \\ 3 & 0 & 3 & 0 & 9 \end{bmatrix}$

Use elementary row operations to row reduce each matrix to row reduced echelon form.

9. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 3 & -5 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 1 & 3 & 1 & 0 \\ 1 & -1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 0 & -2 & 3 & 2 & 1 \\ 0 & 2 & -1 & 4 & 0 \\ 0 & 6 & -7 & 0 & -2 \\ 0 & 4 & -6 & -4 & -2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ 2 & 4 & 0 & 0 & 6 \\ 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

13. $\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ -3 & 1 & 3 & 0 & 1 & 0 \\ 7 & -1 & -4 & 0 & 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ -1 & 2 & 0 \\ 1 & 6 & 8 \\ 0 & 4 & 4 \end{bmatrix}$

15. $\begin{bmatrix} 5 & 1 & 8 & 1 \\ 1 & 1 & 4 & 0 \\ 2 & 0 & 2 & 1 \\ 4 & 1 & 7 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 2 & 8 & 0 & 0 & 6 \\ 1 & 4 & 1 & 1 & 7 \\ -1 & -4 & 0 & 1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 \end{bmatrix}$

Solve the following systems of linear equations:

18. $\begin{array}{rcrcrcr} x & + & 3y & & & = & 2 \\ 5x & & & + & 3z & = & -5 \\ 3x & - & y & + & 2z & = & -4 \end{array}$

19. $\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & + & 9x_3 & + & 8x_4 & = & 10 \\ x_1 & + & & & x_3 & + & 2x_4 & = & 4 \\ -2x_1 & + & x_2 & + & x_3 & - & 3x_4 & = & -9 \\ x_1 & + & x_2 & + & 4x_3 & + & 3x_4 & = & 3 \end{array}$

20. $$\begin{array}{rcrcr} -x & + & 4y & = & -3x \\ x & - & y & = & -3y \end{array}$$

21. $$\begin{array}{rcrcrcrcr} -2x_1 & - & 8x_2 & - & x_3 & - & x_4 & = & -9 \\ -x_1 & - & 4x_2 & & & - & x_4 & = & -8 \\ x_1 & + & 4x_2 & + & x_3 & + & x_4 & = & 6 \end{array}$$

22. $$\begin{array}{rcrcrcr} 2x & + & 3y & + & 8z & = & 5 \\ 2x & + & y & + & 10z & = & 3 \\ 2x & & & + & 8z & = & 4 \end{array}$$

23. $$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & 5x_4 & = & 3 \\ & & x_2 & + & x_3 & + & 4x_4 & = & 1 \\ x_1 & & & + & x_3 & + & 2x_4 & = & 2 \\ 2x_1 & + & 2x_2 & + & 3x_3 & + & 11x_4 & = & 8 \\ 2x_1 & + & x_2 & + & 2x_3 & + & 7x_4 & = & 7 \end{array}$$

24. $$\begin{array}{rcrcrcrcrcrcr} x_1 & + & x_2 & & & = & 3 & + & x_1 & & & & \\ & & x_2 & + & 2x_3 & = & 4 & & & + & x_2 & + & x_3 \\ x_1 & + & 3x_2 & + & 4x_3 & = & 11 & + & x_1 & + & 2x_2 & + & 2x_3 \end{array}$$

25. Suppose the homogeneous system $A\mathbf{x} = \mathbf{0}$ has the following two solutions: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Is $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ a solution? Why or why not?

26. For what value of k will the following system have a solution:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 4 \\ x_1 & - & 2x_2 & + & 8x_3 & = & k \end{array}$$

27. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 7 \\ 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$,and $\mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Solve $A\mathbf{x} = \mathbf{b}_i$, for each $i = 1, 2, 3$.

   (b) Solve the above systems simultaneously by row reducing

$$[A|b_1|b_2|b_3] = \left[\begin{array}{rrr|rrr} 1 & 3 & 4 & 1 & 1 & 1 \\ -2 & 1 & 7 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

5.3 Invertible Matrices

Let A be a square matrix. A matrix B is said to be an **inverse** of A if $BA = AB = I$. In this case we say A is **invertible** or **nonsingular**. If A is not invertible we say A is **singular**.

Example 5.3.1. Suppose

$$A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}.$$

Show that A is invertible and an inverse is

$$B = \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}.$$

► **Solution.** Observe that

$$AB = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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The following proposition says that when A has an inverse there can only be one.

Proposition 5.3.2. *Let A be an invertible matrix. Then the inverse is unique.*

Proof. Suppose B and C are inverses of A. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

□

Because of uniqueness we can properly say **the inverse** of A when A is invertible. In Example 5.3.1, the matrix $B = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$ is the inverse of A; there are no others. It is standard convention to denote the inverse of A by A^{-1}.

For many matrices it is possible to determine their inverse by inspection. For example, the identity matrix I_n is invertible and its inverse is I_n: $I_nI_n = I_n$. A diagonal matrix $\mathrm{diag}(a_1, \ldots, a_n)$ is invertible if each $a_i \neq 0$, $i = 1, \ldots, n$. The inverse then is simply $\mathrm{diag}(\frac{1}{a_1}, \ldots, \frac{1}{a_n})$. However, if one of the a_i is zero then the matrix in not invertible. Even more is true. If A has a zero row, say the i^{th} row, then A is not invertible. To see this we get from Equation (2) in Section 5.1 that $\mathrm{Row}_i(AB) = \mathrm{Row}_i(A)B = \mathbf{0}$. Hence, there is no matrix B for which $AB = I$. Similarly, a matrix with a zero column cannot be invertible.

Proposition 5.3.3. *Let A and B be invertible matrices. Then*

1. *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*
2. *AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof. Suppose A and B are invertible. The symmetry of the equation $A^{-1}A = AA^{-1} = I$ says that A^{-1} is invertible and $(A^{-1})^{-1} = A$. Also $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$ and $(AB)(B^{-1}A^{-1}) = A(B^{-1}B)A^{-1} = AA^{-1} = I$. This shows $(AB)^{-1} = B^{-1}A^{-1}$. □

The following corollary easily follows:

Corollary 5.3.4. *If $A = A_1 \cdots A_k$ is the product of invertible matrices then A is invertible and $A^{-1} = A_k^{-1} \cdots A_1^{-1}$.*

Inversion Computations

Let $\mathbf{e}_i$ be the column vector with 1 in the i^{th} position and 0's elsewhere. By Equation (1) of Section 5.1 the equation $AB = I$ implies that $A\,\mathrm{Col}_i(B) = \mathrm{Col}_i(I) = \mathbf{e}_i$. This means that the solution to $A\mathbf{x} = \mathbf{e}_i$ is the i^{th} column of the inverse of A, when A is invertible. We can thus compute the inverse of A one column at a time using the Gauss-Jordan elimination method on the augmented matrix $[A|\mathbf{e}_i]$. Better yet, though, is to perform the Gauss-Jordan elimination method on the matrix $[A|I]$. If A is invertible it will reduce to a matrix of the form $[I|B]$ and B will be A^{-1}. If A is not invertible it will not be possible to produce the identity in the first slot.

We illustrate this in the following two examples.

Example 5.3.5. Determine whether the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix}$$

is invertible. If it is compute the inverse.

▶ **Solution.** We will augment A with I and follow the procedure outlined above:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[p_{1\,3}]{t_{1\,3}(-1)} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 & 0 & 0 \end{array}\right] \xrightarrow[t_{2\,3}(-2)]{t_{1\,3}(-2)}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 3 & -2 & -2 \end{array}\right] \xrightarrow[\substack{t_{3\,2}(-1)\\ t_{3\,1}(-1)}]{m_3(-1)} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -3 & 2 & 2 \end{array}\right] \xrightarrow{t_{2\,1}(1)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -3 & -3 \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -3 & 2 & 2 \end{array}\right].$$

It follows that A is invertible and $A^{-1} = \begin{bmatrix} 5 & -3 & -3 \\ 3 & -1 & -2 \\ -3 & 2 & 2 \end{bmatrix}$. ◀

Example 5.3.6. Let $A = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 1 & 3 \\ 0 & -7 & 3 \end{bmatrix}$. Determine whether A is invertible. If it is find its inverse.

▶ **Solution.** Again, we augment A with I and row reduce:

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 9 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[t_{2\,3}(-1)]{t_{1\,2}(-2)} \left[\begin{array}{ccc|ccc} 1 & -4 & 0 & 1 & 0 & 0 \\ 0 & 9 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 \end{array}\right]$$

We can stop at this point. Notice that the row operations produced a **0** row in the reduction of A. This implies A cannot be invertible. ◀

Solving a System of Equations

Suppose A is a square matrix with a known inverse. Then the equation $A\mathbf{x} = \mathbf{b}$ implies $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$ and thus gives the solution.

Example 5.3.7. Solve the following system:

$$\begin{array}{rcrcrcl} 2x & + & & + & 3z & = & 1 \\ & & y & + & z & = & 2 \\ 3x & - & y & + & 4z & = & 3. \end{array}$$

▶ **Solution.** The coefficient matrix is

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix}$$

whose inverse we computed in the example above:

$$A^{-1} = \begin{bmatrix} 5 & -3 & -3 \\ 3 & -1 & -2 \\ -3 & 2 & 2 \end{bmatrix}.$$

The solution to the system is thus

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 5 & -3 & -3 \\ 3 & -1 & -2 \\ -3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 \\ -5 \\ 7 \end{bmatrix}.$$

◀

Exercises

Determine whether the following matrices are invertible. If so, find the inverse:

1. $\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -3 \\ 2 & 5 & 5 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & -2 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 2 & -2 & 0 \\ 1 & -1 & 0 & 4 \\ 1 & 2 & 3 & 9 \end{bmatrix}$

10. $\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

12. $\begin{bmatrix} -3 & 2 & -8 & 2 \\ 0 & 2 & -3 & 5 \\ 1 & 2 & 3 & 5 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

Solve each system $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ are given below, by first computing A^{-1} and and applying it to $A\mathbf{x} = \mathbf{b}$ to get $\mathbf{x} = A^{-1}\mathbf{b}$.

13. $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

16. $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 5 & -2 \\ - & 2 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

17. $A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 2 & -2 & 0 \\ 1 & -1 & 0 & 4 \\ 1 & 2 & 3 & 9 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$

18. $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}$

19. Suppose A is an invertible matrix. Show that A^t is invertible and give a formula for the inverse.

20. Let $E(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Show $E(\theta)$ is invertible and find its inverse.

21. Let $F(\theta) = \begin{bmatrix} \sinh\theta & \cosh\theta \\ \cosh\theta & \sinh\theta \end{bmatrix}$. Show $F(\theta)$ is invertible and find its inverse.

22. Suppose A is invertible and $AB = AC$. Show that $B = C$. Give an example of a nonzero matrix A (not invertible) with $AB = AC$, for some B and C, but $B \neq C$.

5.4 Determinants

In this section we will discuss the definition of the determinant and some of its properties. For our purposes the determinant is a very useful number that we can associate to a square matrix. The determinant has an wide range of applications. It can be used to determine whether a matrix is invertible. Cramer's rule gives the unique solution to a system of linear equations as the quotient of determinants. In multidimensional calculus, the Jacobian is given by a determinant and expresses how area or volume changes under a transformation. Most students by now are familiar with the definition of the determinant for a 2×2 matrix: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The **determinant** of A is given by

$$\det(A) = ad - bc.$$

It is the product of the diagonal entries minus the product of the off diagonal entries. For example, $\det \begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix} = 1 \cdot (-2) - 5 \cdot 3 = -17.$

The definition of the determinant for an $n \times n$ matrix is decidedly more complicated. We will present an inductive definition. Let A be an $n \times n$ matrix and let $A(i, j)$ be the matrix obtained from A by deleting the i^{th} row and j^{th} column. Since $A(i, j)$ is an $(n-1) \times (n-1)$ matrix we can inductively define the (i, j) **minor**, $\text{Minor}_{i\,j}(A)$, to be the determinant of $A(i, j)$:

$$\text{Minor}_{i\,j}(A) = \det(A(i, j)).$$

The following theorem, whose proof is extremely tedious and we omit, is the basis for the definition of the determinant.

Theorem 5.4.1 (Laplace expansion formulas). *Suppose A is an $n \times n$ matrix. Then the following numbers are all equal and we call this number the* **determinant of A***:*

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \text{Minor}_{i\,j}(A) \quad \textit{for each } i$$

and

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \text{Minor}_{i\,j}(A) \quad \textit{for each } j.$$

Any of these formulas can thus be taken as the definition of the determinant. In the first formula the index i is fixed and the sum is taken over all j. The entries $a_{i,j}$ thus fill out the i^{th} row. We therefore call this formula the **Laplace expansion of the determinant along the i^{th} row** or simply a **row expansion** . Since the index i can range from 1 to n there are n row expansions. In a similar way, the second formula is called the **Laplace expansion of the determinant along the j^{th} column** or simply a **column expansion** and there are n column expansions. The presence of the factor $(-1)^{i+j}$ alternates the signs along the row or column according as $i + j$ is even or odd.

The **sign matrix**

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is a useful tool to organize the signs in an expansion.

It is common to use the absolute value sign $|A|$ to denote the determinant of A. This should not cause confusion unless A is a 1×1 matrix, in which case we will not use this notation.

Example 5.4.2. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & -2 & 4 \\ 1 & 0 & 5 \end{bmatrix}.$$

▶ **Solution.** For purposes of illustration we compute the determinant in two ways. First, we expand along the first row.

$$\det A = 1 \cdot \begin{vmatrix} -2 & 4 \\ 0 & 5 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} + (-2) \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = 1 \cdot (-10) - 2 \cdot (11) - 2(2) = -36.$$

Second, we expand along the second column.

$$\det A = (-)2 \begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} + (-2) \begin{vmatrix} 1 & -2 \\ 1 & 5 \end{vmatrix} (-)0 \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} = (-2) \cdot 11 - 2 \cdot (7) = -36.$$

Of course, we get the same answer; that's what the theorem guarantees. Observe though that the second column has a zero entry which means that we really only needed to compute two minors. In practice we usually try to use an expansion along a row or column that has a lot of zeros. Also note that we use the sign matrix to adjust the signs on the appropriate terms. ◀

Properties of the determinant

The determinant has many important properties. The three listed below show how the elementary row operations effect the determinant. They are used extensively to simplify many calculations.

Corollary 5.4.3. *Let A be an $n \times n$ matrix. Then*

1. $\det p_{i,j} A = -\det A$.

2. $\det m_i(a) A = a \det A$.

3. $\det t_{i,j}(a) = \det A$.

Proof. We illustrate the proof for the 2×2 case. Let $A = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$. We then have

1. $|p_{1,2}(A)| = \begin{vmatrix} t & u \\ r & s \end{vmatrix} = ts - ru = -|A|$.

2. $|t_{1,2}(a)(A)| = \begin{vmatrix} r & s \\ t + ar & u + as \end{vmatrix} = r(u + as) - s(t + ar) = |A|$.

3. $|m_1(a)(A) = \begin{vmatrix} ar & as \\ t & u \end{vmatrix} = aru - ast = a|A|$.

□

Further important properties include:

1. If A has a zero row (or column) then $\det A = 0$.

2. If A has two equal rows (or columns) then $\det A = 0$.

3. $\det A = \det A^t$.

Example 5.4.4. Use elementary row operations to find $\det A$ if

$$1)\ A = \begin{bmatrix} 2 & 4 & 2 \\ -1 & 3 & 5 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad 2)\ A = \begin{bmatrix} 1 & 0 & 5 & 1 \\ -1 & 2 & 1 & 3 \\ 2 & 2 & 16 & 6 \\ 3 & 1 & 0 & 1 \end{bmatrix}.$$

► **Solution.** Again we will write the elementary row operation that we have used above the equal sign.

$$1)\quad \begin{vmatrix} 2 & 4 & 2 \\ -1 & 3 & 5 \\ 0 & 1 & 1 \end{vmatrix} \overset{m_1(\frac{1}{2})}{=} 2\begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & 5 \\ 0 & 1 & 1 \end{vmatrix} \overset{t_{12}(1)}{=} 2\begin{vmatrix} 1 & 2 & 1 \\ 0 & 5 & 6 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\overset{p_{23}}{=} -2\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 6 \end{vmatrix} \overset{t_{23}(-5)}{=} -2\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -2.$$

In the last equality we have used the fact that the last matrix is upper triangular and its determinant is the product of the diagonal entries.

$$2)\quad \begin{vmatrix} 1 & 0 & 5 & 1 \\ -1 & 2 & 1 & 3 \\ 2 & 2 & 16 & 6 \\ 3 & 1 & 0 & 1 \end{vmatrix} \overset{\substack{t_{12}(1)\\ t_{13}(-2)\\ t_{14}(-3)}}{=} \begin{vmatrix} 1 & 0 & 5 & 1 \\ 0 & 2 & 6 & 4 \\ 0 & 2 & 6 & 4 \\ 0 & 1 & -15 & -2 \end{vmatrix} = 0,$$

because two rows are equal. ◄

In the following example we use elementary row operations to zero out entries in a column and then use a Laplace expansion formula.

Example 5.4.5. Find the determinant of

$$A = \begin{bmatrix} 1 & 4 & 2 & -1 \\ 2 & 2 & 3 & 0 \\ -1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \end{bmatrix}.$$

► **Solution.**

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 4 & 2 & -1 \\ 2 & 2 & 3 & 0 \\ -1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \end{vmatrix} \overset{\substack{t_{1,2}(-2)\\ t_{1,3}(1)}}{=} \begin{vmatrix} 1 & 4 & 2 & -1 \\ 0 & -6 & -1 & 2 \\ 0 & 5 & 4 & 3 \\ 0 & 1 & 3 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -6 & -1 & 2 \\ 5 & 4 & 3 \\ 1 & 3 & 2 \end{vmatrix} \overset{\substack{t_{3,1}(6)\\ t_{3,2}(-5)}}{=} \begin{vmatrix} 0 & 17 & 14 \\ 0 & -11 & -7 \\ 1 & 3 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 17 & 14 \\ -11 & -7 \end{vmatrix} = -119 + 154 = 35 \end{aligned}$$

◀

The following theorem contains two very important properties of the determinant. We will omit the proof.

Theorem 5.4.6.

1. *A square matrix* A *is invertible if and only if* $\det A \neq 0$.
2. *If* A *and* B *are square matrices of the same size then*

$$\det(AB) = \det A \det B.$$

The cofactor and adjoint matrices

Again, let A be a square matrix. We define the **cofactor** matrix, $\mathrm{Cof}(A)$, of A to be the matrix whose (i,j)-entry is $(-1)^{i+j}\mathrm{Minor}_{i,j}$. We define the **adjoint** matrix, $\mathrm{Adj}(A)$, of A by the formula $\mathrm{Adj}(A) = (\mathrm{Cof}(A))^t$. The important role of the adjoint matrix is seen in the following theorem and its corollary.

Theorem 5.4.7. *For* A *a square matrix we have*

$$A\ \mathrm{Adj}(A) = \mathrm{Adj}(A)\ A = \det(A)I.$$

Proof. The (i,j) entry of $A\ \mathrm{Adj}(A)$ is

$$\sum_{k=0}^{n} A_{i\,k}(\mathrm{Adj}(A))_{k\,j} = \sum_{k=0}^{n}(-1)^{k+j}A_{i\,k}\mathrm{Minor}_{k\,j}(A).$$

When $i = j$ this is a Laplace expansion formula and is hence $\det A$ by Theorem 5.4.1. When $i \neq j$ this is the expansion of a determinant for a matrix with two equal rows and hence is zero. □

The following corollary immediately follows.

Corollary 5.4.8 (The adjoint inversion formula). *If* $\det A \neq 0$ *then*

$$A^{-1} = \frac{1}{\det A}\mathrm{Adj}(A).$$

The inverse of a 2×2 matrix is a simple matter: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\mathrm{Adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and if $\det(A) = ad - bd \neq 0$ then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1}$$

For an example suppose $A = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$. Then $\det(A) = 1 - (6) = -5 \neq 0$ so A is invertible and $A^{-1} = \frac{-1}{5} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{-1}{5} \end{bmatrix}$.

The general formula for the inverse of a 3×3 is substantially more complicated and difficult to remember. Consider though an example.

Example 5.4.9. Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 4 & 1 \\ -1 & 0 & 3 \end{bmatrix}.$$

Find its inverse if it is invertible.

► **Solution.** We expand along the first row to compute the determinant and get $\det(A) = 1 \det \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} - 2 \det \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = 1(12) - 2(4) = 4$. Thus A is invertible. The cofactor of A is $\mathrm{Cof}(A) = \begin{bmatrix} 12 & -4 & 4 \\ -6 & 3 & -2 \\ 2 & -1 & 2 \end{bmatrix}$ and $\mathrm{Adj}(A) = \mathrm{Cof}(A)^t = \begin{bmatrix} 12 & -6 & 2 \\ -4 & 3 & -1 \\ 4 & -2 & 2 \end{bmatrix}$. The inverse of A is thus

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 12 & -6 & 2 \\ -4 & 3 & -1 \\ 4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & \frac{-3}{2} & \frac{1}{2} \\ -1 & \frac{3}{4} & \frac{-1}{4} \\ 1 & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}.$$

◄

In our next example we will consider a matrix with entries in $\mathcal{R} = \mathbb{R}[s]$. Such matrices will arise naturally in Chapter 6.

Example 5.4.10. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

Find the inverse of the matrix

$$sI - A = \begin{bmatrix} s-1 & -2 & -1 \\ 0 & s-1 & -3 \\ -1 & -1 & s-2 \end{bmatrix}$$

.

▶ **Solution.** A straightforward computation gives $\det(sI - A) = (s-4)(s^2+1)$. The matrix of minors for $sI - A$ is

$$\begin{bmatrix} (s-1)(s-2)-3 & -3 & s-1 \\ -2(s-2)-1 & (s-1)(s-2)-1 & -(s-1)-2 \\ 6+(s-1) & -3(s-1) & (s-1)^2 \end{bmatrix}.$$

After simplifying somewhat we obtain the cofactor matrix

$$\begin{bmatrix} s^2-3s-1 & 3 & s-1 \\ 2s-3 & s^2-3s+1 & s+1 \\ s+5 & 3s-3 & (s-1)^2 \end{bmatrix}.$$

The adjoint matrix is

$$\begin{bmatrix} s^2-3s-1 & 2s-3 & s+5 \\ 3 & s^2-3s+1 & 3s-3 \\ s-1 & s+1 & (s-1)^2 \end{bmatrix}.$$

Finally, we obtain the inverse:

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s^2-3s-1}{(s-4)(s^2+1)} & \frac{2s-3}{(s-4)(s^2+1)} & \frac{s+5}{(s-4)(s^2+1)} \\ \frac{3}{(s-4)(s^2+1)} & \frac{s^2-3s+1}{(s-4)(s^2+1)} & \frac{3s-3}{(s-4)(s^2+1)} \\ \frac{s-1}{(s-4)(s^2+1)} & \frac{s+1}{(s-4)(s^2+1)} & \frac{(s-1)^2}{(s-4)(s^2+1)} \end{bmatrix}.$$

◀

Cramer's Rule

We finally consider a well known theoretical tool used to solve a system $A\mathbf{x} = \mathbf{b}$ when A is invertible. Let $A(i, \mathbf{b})$ denote the matrix obtained by replacing the i^{th} column of A with the column vector $\mathbf{b}$. We then have the following theorem:

Theorem 5.4.11. *Suppose* $\det A \neq 0$. *Then the solution to* $A\mathbf{x} = \mathbf{b}$ *is given coordinate wise by the formula:*

$$\mathbf{x}_i = \frac{\det A(i, \mathbf{b})}{\det A}.$$

Proof. Since A is invertible we have

$$\begin{aligned} \mathbf{x}_i &= (A^{-1}\mathbf{b})_i = \sum_{k=1}^{n} (A^{-1})_{i\,k}\mathbf{b}_k \\ &= \frac{1}{\det A}\sum_{k=1}^{n}(-1)^{i+k}\mathrm{Minor}_{k\,i}(A)\mathbf{b}_k \\ &= \frac{1}{\det(A)}\sum_{k=1}^{n}(-1)^{i+k}\mathbf{b}_k\mathrm{Minor}_{k\,i}(A) = \frac{\det A(i,\mathbf{b})}{\det A}. \end{aligned}$$

□

The following example should convince you that Cramer's Rule is mainly a theoretical tool and not a practical one for solving a system of linear equations. The Gauss-Jordan elimination method is usually far more efficient than computing $n + 1$ determinants for a system $A\mathbf{x} = \mathbf{b}$, where A is $n \times n$.

Example 5.4.12. Solve the following system of linear equations using Cramer's Rule.

$$\begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ 2x & + & 3y & - & z & = & 11 \\ x & & & + & z & = & -2 \end{array}$$

▶ **Solution.** We have

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -3,$$

$$\det A(1,\mathbf{b}) = \begin{vmatrix} 0 & 1 & 1 \\ 11 & 3 & -1 \\ -2 & 0 & 1 \end{vmatrix} = -3,$$

$$\det A(2,\mathbf{b}) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 11 & -1 \\ 1 & -2 & 1 \end{vmatrix} = -6,$$

$$\text{and } \det A(3,\mathbf{b}) = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 3 & 11 \\ 1 & 0 & -2 \end{vmatrix} = 9,$$

where $\mathbf{b} = \begin{bmatrix} 0 \\ 11 \\ -2 \end{bmatrix}$. Since $\det A \neq 0$ Cramer's Rule gives

$$x_1 = \frac{\det A(1,\mathbf{b})}{\det A} = \frac{-3}{-3} = 1,$$

$$x_2 = \frac{\det A(2,\mathbf{b})}{\det A} = \frac{-6}{-3} = 2,$$

and

$$x_3 = \frac{\det A(3,\mathbf{b})}{\det A} = \frac{9}{-3} = -3.$$

◀

Exercises

Find the determinant of each matrix given below in three ways: a row expansion, a column expansion, and using row operations to reduce to a triangular matrix.

1. $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 4 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 0 & 3 \\ 8 & 1 & 7 \\ 3 & 4 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 3 & 98 & 100 \\ 0 & 2 & 99 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 2 & 3 & 9 & 2 \\ 1 & 4 & 8 & 3 \\ -2 & 3 & -2 & 4 \end{bmatrix}$

8. $\begin{bmatrix} -4 & 9 & -4 & 1 \\ 2 & 3 & 0 & -4 \\ -2 & 3 & 5 & -6 \\ -3 & 2 & 0 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 4 & 2 & 3 \\ 1 & 2 & 1 & 4 \\ 4 & 8 & 4 & 6 \\ 1 & 9 & 11 & 13 \end{bmatrix}$

Find the inverse of $(sI - A)$ and determine for which values of s $\det(sI - A) = 0$.

10. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -3 & 3 \\ -3 & 1 & 3 \\ 3 & -3 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{bmatrix}$

Use the adjoint formula for the inverse for the matrices given below.

16. $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$

18. $\begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 4 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 4 & 0 & 3 \\ 8 & 1 & 7 \\ 3 & 4 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 98 & 100 \\ 0 & 2 & 99 \\ 0 & 0 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 2 & 3 & 9 & 2 \\ 1 & 4 & 8 & 3 \\ -2 & 3 & -2 & 4 \end{bmatrix}$

23. $\begin{bmatrix} -4 & 9 & -4 & 1 \\ 2 & 3 & 0 & -4 \\ -2 & 3 & 5 & -6 \\ -3 & 2 & 0 & 1 \end{bmatrix}$

24. $\begin{bmatrix} 2 & 4 & 2 & 3 \\ 1 & 2 & 1 & 4 \\ 4 & 8 & 4 & 6 \\ 1 & 9 & 11 & 13 \end{bmatrix}$

Chapter 6

SYSTEMS OF DIFFERENTIAL EQUATIONS

6.1 Systems of Differential Equations

6.1.1 Introduction

In the previous chapters we have discussed ordinary differential equations in a single unknown function. These are adequate to model real world systems as they evolve in time, provided that only one state, i.e., one number $y(t)$, is necessary to describe the system. For instance, we might be interested in the way that the population of a species changes over time, the way the temperature of an object changes over time, the way the concentration of a pollutant in a lake changes over time, or the displacement over time of a weight attached to a spring. In each of these cases, the system we wish to describe is adequately represented by a single number. In the examples listed, the number is the population $p(t)$ at time t, the temperature $T(t)$ at time t, the concentration $c(t)$ of a pollutant at time t, or the displacement $y(t)$ of the weight from equilibrium. However, a single ordinary differential equation is inadequate for describing the evolution over time of a system which needs more than one number to describe its state at a given time t. For example, an ecological system consisting of two species will require two numbers $p_1(t)$ and $p_2(t)$ to describe the population of each species at time t, i.e., to describe a system consisting of a population of rabbits and foxes, you need to give the population of both rabbits and foxes at time t. Moreover, the description of the way this system changes with time will involve the derivatives $p_1'(t)$, $p_2'(t)$, the functions $p_1(t)$, $p_2(t)$ themselves, and possibly the variable t. This is precisely what is intended by a system of ordinary differential equations.

A **system of ordinary differential equations** is a system of equations relating several unknown functions $y_i(t)$ of an independent variable t, some of the derivatives of the $y_i(t)$, and possibly t itself. As for a single differential equation, the **order** of a system of differential equations is the highest order derivative which appears in any equation.

Example 6.1.1. The following two equations

$$\begin{array}{rcrcl} y_1' &=& ay_1 &-& by_1y_2 \\ y_2' &=& -cy_1 &+& dy_1y_2 \end{array} \tag{1}$$

constitute a system of ordinary differential equations involving the unknown functions y_1 and y_2. Note that in this example the number of equations is equal to the number of unknown functions. This is the typical situation which occurs in practice.

Example 6.1.2. Suppose that a particle of mass m moves in a force field $\mathbf{F} = (F_1, F_2, F_3)$ that depends on time t, the position of the particle $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$ and the velocity of the particle $\mathbf{x}'(t) = (x_1'(t), x_2'(t), x_3'(t))$. Then Newton's second law of motion states, in vector form, that $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a} = \mathbf{x}''$ is the acceleration. Writing out what this says in components, we get a system of second order differential equations

$$\begin{array}{rcl} mx_1''(t) &=& F_1(t, x_1(t), x_2(t), x_3(t), x_1'(t), x_2'(t), x_3'(t)) \\ mx_2''(t) &=& F_2(t, x_1(t), x_2(t), x_3(t), x_1'(t), x_2'(t), x_3'(t)) \\ mx_3''(t) &=& F_3(t, x_1(t), x_2(t), x_3(t), x_1'(t), x_2'(t), x_3'(t)). \end{array} \tag{2}$$

In this example, the state at time t is described by six numbers, namely the three coordinates and the three velocities, and these are related by the three equations described above. The resulting system of equations is a *second order system* of differential equations since the equations include second order derivatives of some of the unknown functions. Notice that in this example we have six states, namely the three coordinates of the position vector and the three coordinates of the velocity vector, but only three equations. Nevertheless, it is easy to put this system in exactly the same theoretical framework as the first example by renaming the states as follows. Let $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5, y_6)$ where $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$, $y_4 = x_1'$, $y_5 = x_2'$, and $y_6 = x_3'$. Using these new function names, the system of equations (2) can be rewritten using only first derivatives:

$$\begin{array}{rcl} y_1' &=& y_4 \\ y_2' &=& y_5 \\ y_3' &=& y_6 \\ y_4' &=& \dfrac{1}{m}F_1(t, y_1, y_2, y_3, y_4, y_5, y_6) \\ y_5' &=& \dfrac{1}{m}F_2(t, y_1, y_2, y_3, y_4, y_5, y_6) \\ y_6' &=& \dfrac{1}{m}F_3(t, y_1, y_2, y_3, y_4, y_5, y_6). \end{array} \tag{3}$$

Note that this can be expressed as a vector equation

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$$

where

$$\mathbf{f}(t, \mathbf{y}) = (y_4, y_5, y_6, \frac{1}{m}F_1(t, \mathbf{y}), \frac{1}{m}F_2(t, \mathbf{y}), \frac{1}{m}F_3(t, \mathbf{y})).$$

The trick used in Example 6.1.2 to reduce the second order system to a first order system in a larger number of variables works in general, so that it is only really necessary to consider first order systems of differential equations.

As with a single first order ordinary differential equation, it is convenient to consider first order systems in a standard form for purposes of describing properties and solution algorithms for these systems.

Definition 6.1.3. The **standard form** for a first order system of ordinary differential equations is a vector equation of the form

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \tag{4}$$

where $\mathbf{f} : U \to \mathbb{R}^n$ is a function from an open subset U of $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$. If an initial point t_0 and an initial vector $\boldsymbol{y}_0$ are also specified, then one obtains an **initial value problem**:

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0. \tag{5}$$

A **solution** of Equation (4) is a differentiable vector function $\mathbf{y} : I \to \mathbb{R}^n$ where I is an open interval in $\mathbb{R}$ and the function $\mathbf{y}$ satisfies Equation (4) for all $t \in I$. This means that

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t)) \tag{6}$$

for all $t \in I$. If also $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$, then $\boldsymbol{y}(t)$ is a solution of the initial value problem (5).

Equation (1) is a system in standard form where $n = 2$. That is, there are two unknown functions y_1 and y_2 which can be incorporated into a two dimensional vector $\boldsymbol{y} = (y_1,\, y_2)$, and if $\mathbf{f}(t, \mathbf{y}) = (f_1(t, y_1, y_2), f_2(t, y_{1,2}\,)) = (ay_1 - by_1y_2,\ -cy_1 + dy_1y_2)$, then Equation (5) is a short way to write the system of equations

$$\begin{aligned} y_1' &= ay_1 - by_1y_2 &&= f_1(t, y_1, y_2) \\ y_2' &= -cy_1 + dy_1y_2 &&= f_2(t, y_1, y_2). \end{aligned} \tag{7}$$

Equation (3) of Example 6.1.2 is a first order system with $n = 6$. We shall primarily concentrate on the study of systems where $n = 2$ or $n = 3$, but Example 6.1.2 shows that even very simple real world systems can lead to systems of differential equations with a large number of unknown functions.

Example 6.1.4. Consider the following first order system of ordinary differential equations:

$$\begin{aligned} y_1' &= 3y_1 - y_2 \\ y_2' &= 4y_1 - 2y_2. \end{aligned} \tag{8}$$

1. Verify that $\boldsymbol{y}(t) = (y_1(t),\, y_2(t)) = (e^{2t},\, e^{2t})$ is a solution of Equation (8).

   ▶ **Solution.** Since $y_1(t) = y_2(t) = e^{2t}$,

   $$\begin{aligned} &y_1'(t) = 2e^{2t} = 3e^{2t} - e^{2t} = 3y_1(t) - y_2(t) \\ \text{and}\quad &y_2'(t) = 2e^{2t} = 4e^{2t} - 2e^{2t} = 4y_1(t) - 2y_2(t), \end{aligned}$$

   which is precisely what it means for $\boldsymbol{y}(t)$ to satisfy (8). ◀

2. Verify that $\boldsymbol{z}(t) = (z_1(t),\, z_2(t)) = (e^{-t},\, 4e^{-t})$ is a solution of Equation (8).

   ▶ **Solution.** As above, we calculate

   $$\begin{aligned} &z_1'(t) = -e^{-t} = 3e^{-t} - 4e^{-t} = 3z_1(t) - z_2(t) \\ \text{and}\quad &z_2'(t) = -4e^{-t} = 4e^{-t} - 2\cdot 4e^{-t} = 4z_1(t) - 2z_2(t), \end{aligned}$$

   which is precisely what it means for $\boldsymbol{z}(t)$ to satisfy (8). ◀

3. If c_1 and c_2 are *any* constants, verify that $\boldsymbol{w}(t) = c_1\boldsymbol{y}(t) + c_2\boldsymbol{z}(t)$ is also a solution of Equation (8).

   ▶ **Solution.** Note that $\boldsymbol{w}(t) = (w_1(t),\, w_2(t))$, where $w_1(t) = c_1y_1(t) + c_2z_1(t) = c_1e^{2t} + c_2e^{-t}$ and $w_2(t) = c_1y_2(t) + c_2z_2(t) = c_1e^{2t} + c_24e^{-t}$. Then

   $$\begin{aligned} &w_1'(t) = 2c_1e^{2t} - c_2e^{-t} = 3w_1(t) - w_2(t) \\ \text{and}\quad &w_2'(t) = 2c_1e^{-t} - 4c_2e^{-t} = 4w_1(t) - 2w_2(t). \end{aligned}$$

   Again, this is precisely what it means for $\boldsymbol{w}(t)$ to be a solution of (8). We shall see in the next section that $\boldsymbol{w}(t)$ is, in fact, the general solution of Equation (8). That is, any solution of this equation is obtained by a particular choice of the constants c_1 and c_2. ◀

Example 6.1.5. Consider the following first order system of ordinary differential equations:

$$\begin{aligned} y_1' &= 3y_1 - y_2 + 2t \\ y_2' &= 4y_1 - 2y_2 + 2. \end{aligned} \tag{9}$$

Notice that this is just Equation (8) with one additional term (not involving the unknown functions y_1 and y_2) added to each equation.

1. Verify that $\boldsymbol{y}_p(t) = (y_{p1}(t), y_{p2}(t)) = (-2t+1,\ -4t+5)$ is a solution of Equation (9).

   ▶ **Solution.** Since $y_{p1}(t) = -2t+1$ and $y_{p2}(t) = -4t+5$, direct calculation gives $y'_{p1}(t) = -2$, $y'_{p2}(t) = -4$ and

   $$\begin{aligned} &3y_{p1}(t) - y_{p2}(t) + 2t = 3(-2t+1) - (-4t+5) + 2t = -2 = y'_{p1}(t) \\ \text{and}\quad &4y'_{p1}(t) - 2y'_{p2}(t) + 2 = 4(-2t+1) - 2(-4t+5) + 2 = -4 = y'_{p2}(t). \end{aligned}$$

   Hence $\boldsymbol{y}_p(t)$ is a solution of (9). ◀

2. Verify that $\boldsymbol{z}_p(t) = 2\boldsymbol{y}_p(t) = (z_{p1}(t), z_{p2}(t)) = (-4t+2,\ -8t+10)$ is *not* a solution to Equation (9).

   ▶ **Solution.** Since

   $$3z_{p1}(t) - z_{p2}(t) + 2t = 3(-4t+2) - (-8t+10) + 2t = -2t - 4 \neq -4 = z'_{p1}(t),$$

   $\boldsymbol{z}_p(t)$ fails to satisfy the first of the two equations of (9), and hence is not a solution of the system. ◀

3. We leave it as an exercise to verify that $\boldsymbol{y}_g(t) = \boldsymbol{w}(t) + \boldsymbol{y}_p(t)$ is a solution of (9), where $\boldsymbol{w}(t)$ is the general solution of Equation (8) from the previous example.

We will now list some particular classes of first order systems of ordinary differential equations. As for the case of a single differential equation, it is most convenient to identify these classes by describing properties of the right hand side of the equation when it is expressed in standard form.

Definition 6.1.6. The first order system in standard form

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$$

is said to be

1. **autonomous** if $\mathbf{f}(t, \mathbf{y})$ is independent of t;
2. **linear** if $\mathbf{f}(t, \mathbf{y}) = A(t)\mathbf{y} + \mathbf{q}(t)$ where $A(t) = [a_{ij}(t)]$ is an $n \times n$ matrix of functions and $\mathbf{q}(t) = (q_1(t), \ldots, q_n(t))$ is a vector of functions of t;
3. **constant coefficient linear** if $\mathbf{f}(t, \mathbf{y}) = A\mathbf{y} + \mathbf{q}(t)$ where $A = [a_{ij}]$ is an $n \times n$ constant matrix and $\mathbf{q}(t) = (q_1(t), \ldots, q_n(t))$ is a vector of functions of t;

4. **linear and homogeneous** if $\mathbf{f}(t, \mathbf{y}) = A(t)\mathbf{y}$. That is, a system of linear ordinary differential equations is homogeneous provided the term $\mathbf{q}(t)$ is 0.

In the case $n = 2$, a first order system is linear if it can be written in the form

$$\begin{aligned} y_1' &= a(t)y_1 + b(t)y_2 + q_1(t) \\ y_2' &= c(t)y_1 + d(t)y_2 + q_2(t); \end{aligned}$$

this linear system is homogeneous if

$$\begin{aligned} y_1' &= a(t)y_1 + b(t)y_2 \\ y_2' &= c(t)y_1 + d(t)y_2, \end{aligned}$$

and it is a constant coefficient linear system if

$$\begin{aligned} y_1' &= ay_1 + by_2 + q_1(t) \\ y_2' &= cy_1 + dy_2 + q_2(t) \end{aligned}$$

In the first two cases the matrix of functions is $A(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$, while in the third case, the constant matrix is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Notice that the concepts *constant coefficient* and *autonomous* are not identical for linear systems of differential equations. The linear system $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{q}(t)$ is constant coefficient provided all entries of $A(t)$ are constant functions, while it is autonomous if all entries of both $A(t)$ and $\mathbf{q}(t)$ are constants.

Example 6.1.7. 1. The linear system (1) of Example 6.1.1 is autonomous, but not linear.

2. The system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 - \frac{1}{t}y_2 \end{aligned}$$

is linear and homogeneous, but not autonomous.

3. The system

$$\begin{aligned} y_1' &= -y_2 \\ y_2' &= y_1 \end{aligned}$$

is linear, constant coefficient, and homogeneous (and hence autonomous).

4. The system

$$\begin{aligned} y_1' &= -y_2 + 1 \\ y_2' &= y_1 \end{aligned}$$

is linear and autonomous (and hence constant coefficient) but not homogeneous.

5. The system

$$\begin{aligned} y_1' &= -y_2 + t \\ y_2' &= y_1 \end{aligned}$$

is constant coefficient, but not autonomous or homogeneous.

Note that the term *autonomous* applies to both linear and nonlinear systems of ordinary differential equations, while the term *constant coefficient* applies only to linear systems of differential equations, and as the examples show, even for linear systems, the terms constant coefficient and autonomous do not refer to the same systems.

6.1.2 Examples of Linear Systems

In this section we will look at some situations which give rise to systems of ordinary differential equations. Our goal will be to simply set up the differential equations; techniques for solutions will come in later sections.

Example 6.1.8. The first example is simply the observation that a single ordinary differential equation of order n can be viewed as a first order *system* of n equations in n unknown functions. We will do the case for $n = 2$; the extension to $n > 2$ is straightforward. Let

$$y'' = f(t, y, y'), \quad y(t_0) = a, \; y'(t_0) = b \tag{10}$$

be a second order initial value problem. By means of the identification $y_1 = y$, $y_2 = y'$, Equation (10) can be identified with the system

$$\begin{aligned} y_1' &= y_2 & \qquad y_1(t_0) &= a \\ y_2' &= f(t, y_1, y_2) & \qquad y_2(t_0) &= b \end{aligned} \tag{11}$$

For a numerical example, consider the second order initial value problem

$$(*) \qquad y'' - y = t, \quad y(0) = 1, \quad y'(0) = 2.$$

According to Equation (11), this is equivalent to the system

$$(**)\qquad \begin{array}{rcl} y_1' &=& y_2 \\ y_2' &=& y_1 + t \end{array} \qquad \begin{array}{rcl} y_1(0) &=& 1 \\ y_2(0) &=& 2. \end{array}$$

Equation $(*)$ can be solved by the techniques of Chapter 3 to give a solution $y(t) = 2e^t - e^{-t} - t$. The corresponding solution to the system $(**)$ is the vector function

$$\boldsymbol{y}(t) = (y_1(t),\, y_2(t)) = (2e^t - e^{-t} - t,\, 2e^t + e^{-t} - 1),$$

where $y_1(t) = y(t) = 2e^t - e^{-t} - t$ and $y_2(t) = y'(t) = 2e^t + e^{-t} - 1$.

Example 6.1.9 (Predator-Prey System). In Example 1.1.8 we introduced two differential equation models for the growth of population of a single species. These were the Malthusian proportional growth model, given by the differential equation $p' = kp$ (where, as usual, $p(t)$ denotes the population at time t), and the Verhulst model which is governed by the logistic differential equation $p' = c(m-p)p$, where c and m are constants. In this example we will consider an ecological system consisting of two species, where one species, which we will call the prey, is the food source for another species which we will call the predator. For example we could have coyotes (predator) and rabbits (prey) or sharks (predators) and food fish (prey). Let $p_1(t)$ denote the predator population at time t and let $p_2(t)$ denote the prey population at time t. Using some assumptions we may formulate potential equations satisfied by the rates of change of p_1 and p_2. To talk more succinctly, we will assume that the predators are coyotes, and the prey are rabbits. Let us assume that if there are no coyotes then the rabbit population will increase at a rate proportional to the current population, that is $p_2'(t) = ap_2(t)$ where a is a positive constant. Since the coyotes eat the rabbits, we may assume that the rate at which the rabbits are eaten is proportional to the number of contacts between coyotes and rabbits, which we may assume is proportional to $p_1(t)p_2(t)$; this will, of course, have a negative impact upon the rabbit population. Combining the growth rate (from reproduction) and the rate of decline (from being eaten by coyotes), we arrive at $p_2'(t) = ap_2(t) - bp_1(t)p_2(t)$ where b is a positive constant as a formula expressing the rate of change of the rabbit population. A similar reasoning will apply to the coyote population. If no rabbits are present, then the coyote population will die out, and we will assume that this happens at a rate proportional to the current population. Thus $p_1'(t) = -cp_1(t)$ where c is a positive constant is the first approximation. Moreover, the increase in the population of coyotes is dependent upon interactions with their food supply, i.e., rabbits, so a simple assumption would be that the increase is proportional to the number of interactions between coyotes and rabbits, which we can take to be proportional to $p_1(t)p_2(t)$. Thus, combining the two sources of change in the coyote population gives $p_1'(t) = -cp_1(t) + dp_1(t)p_2(t)$. Therefore, the predator and prey populations are governed by the *first order system* of differential equations

$$\begin{array}{rcl} p_1'(t) &=& -cp_1(t) + dp_1(t)p_2(t) \\ p_2'(t) &=& ap_2(t) - bp_1(t)p_2(t). \end{array} \tag{12}$$

If we let $\mathbf{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$ then Equation (12) can be expressed as the vector equation

$$\mathbf{p}'(t) = \mathbf{f}(t, \mathbf{p}(t)), \tag{13}$$

where

$$\mathbf{f}(t, \mathbf{u}) = \begin{bmatrix} -cu_1 + du_1u_2 \\ au_2 - bu_1u_2 \end{bmatrix}$$

and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, which is a more succinct way to write the system (12) for many purposes. We shall have more to say about this system in a later section.

Example 6.1.10 (Mixing problem). Example 1.1.10 considers the case of computing the amount of salt in a tank at time t if a salt mixture is flowing into the tank at a known volume rate and concentration and the well-stirred mixture is flowing out at a known volume rate. What results is a first order linear differential equation for the amount $y(t)$ of salt at time t (Equation (8)). The current example expands upon the earlier example by considering the case of two connected tanks. See Figure 6.1. Tank 1 contains 200

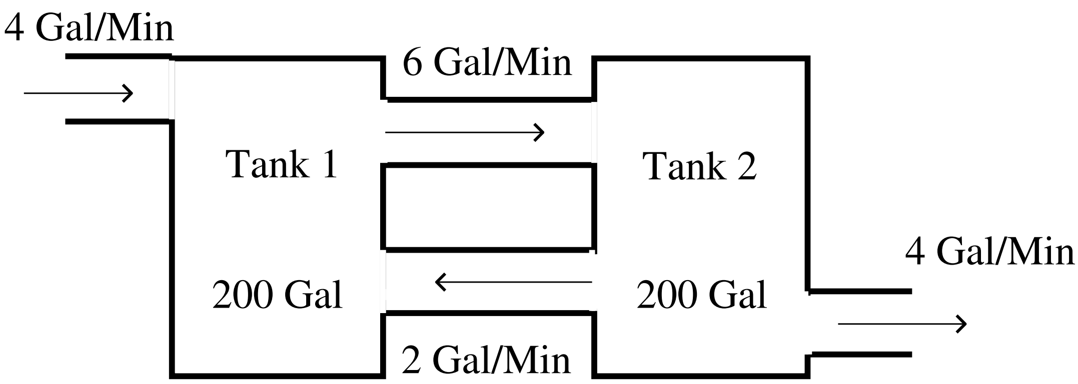

Figure 6.1: A Two Tank Mixing Problem.

gallons of brine which 50 pounds of salt are initially dissolved ; Tank 2 initially contains 200 gallons of pure water. Moreover, the mixtures are pumped between the two tanks, 6 gal/min from Tank 1 to Tank 2 and 2 gal/min going from Tank 2 back to Tank 1. Assume that a brine mixture containing .5 lb/gal enters Tank 1 at a rate of 4 gal/min, and the well-stirred mixture is removed from Tank 2 at the same rate of 4 gal/min. Let $y_1(t)$ be the amount of salt in Tank 1 at time t and let $y_2(t)$ be the amount of salt in Tank 2 at time t. Find a system of differential equations which relates $y_1(t)$ and $y_2(t)$.

► **Solution.** The underlying principle is the same as that of the single tank mixing problem. Namely, we apply the balance equation

$$(\dagger) \qquad y'(t) = \text{Rate in} - \text{Rate out}$$

to the amount of salt in *each* tank. If $y_1(t)$ denotes the amount of salt at time t in Tank 1, then the **concentration** of salt at time t in Tank 1 is $c_1(t) = (y_1(t)/200)$ lb/gal. Similarly, the concentration of salt in Tank 2 at time t is $c_2(t) = (y_2(t)/200)$ lb/gal. The relevant rates of change can be summarized in the following table.

From	**To**	**Rate**		
Outside	Tank 1	(0.5 lb/gal)·(4 gal/min)	=	2 lb/min
Tank 1	Tank2	$\dfrac{y_1(t)}{200} \cdot 6$ gal/min	=	$0.03y_1(t)$ lb/min
Tank 2	Tank 1	$\dfrac{y_2(t)}{200}$lb/gal $\cdot$ 2 gal/min	=	$0.01y_2(t)$ lb/min
Tank 2	Outside	$\dfrac{y_2(t)}{200}$lb/gal $\cdot$ 4 gal/min	=	$0.02y_2(t)$ lb/min

The data for the balance equations (†) can then be read from the following table:

Tank	**Rate in**	**Rate out**
1	$2 + 0.01y_2(t)$	$0.03y_1(t)$
2	$0.03y_1(t)$	$0.02y_2(t)$

Putting these data in the balance equations then gives

$$\begin{aligned} y_1'(t) &= 2 + 0.01y_2(t) - 0.03y_1(t) \\ y_2'(t) &= \phantom{2 +{}} 0.03y_1(t) - 0.02y_2(t) \end{aligned}$$

as the first order system of ordinary differential equations satisfied by the vector function whose two components are the amount of salt in tank 1 and in tank 2 at time t. This system is a nonhomogeneous, constant coefficient, linear system. We shall address some techniques for solving such equations in Section 6.4, after first considering some of the theoretical underpinnings of these equations in the next two sections. ◄

Exercises

For each of the following systems of differential equations, determine if it is linear (yes/no) and autonomous (yes/no). For each of those which is linear, further determine if the equation is homogeneous/nonhomogeneous and constant coefficient (yes/no). Do **not** solve the equations.

1. $\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 y_2 \end{aligned}$

2. $\begin{aligned} y_1' &= y_1 + y_2 + t^2 \\ y_2' &= -y_1 + y_2 + 1 \end{aligned}$

3. $\begin{aligned} y_1' &= (\sin t)y_1 - y_2 \\ y_2' &= y_1 + (\cos t)y_2 \end{aligned}$

4. $\begin{aligned} y_1' &= t\sin y_1 - y_2 \\ y_2' &= y_1 + t\cos y_2 \end{aligned}$

5. $\begin{aligned} y_1' &= y_1 \\ y_2' &= 2y_1 + y_4 \\ y_3' &= y_4 \\ y_4' &= y_2 + 2y_3 \end{aligned}$

6. $\begin{aligned} y_1' &= \frac{1}{2}y_1 - y_2 + 5 \\ y_2' &= -y_1 + \frac{1}{2}y_2 - 5 \end{aligned}$

7. Verify that $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, where $y_1(t) = e^t - e^{3t}$ and $y_2(t) = 2e^t - e^{3t}$ is a solution of the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \boldsymbol{y}; \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution: First note that $y_1(0) = 0$ and $y_2(0) = 1$, so the initial condition is satisfied. Then $\boldsymbol{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} e^t - 3e^{3t} \\ 2e^t - 3e^{3t} \end{bmatrix}$ while $\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \boldsymbol{y}(t) = \begin{bmatrix} 5(e^t - e^{3t}) - 2(2e^t - e^{3t}) \\ 4(e^t - e^{3t}) - (2e^t - e^{3t}) \end{bmatrix} = \begin{bmatrix} e^t - 3e^{3t} \\ 2e^t - 3e^{3t} \end{bmatrix}$. Thus $\boldsymbol{y}'(t) = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \boldsymbol{y}$, as required.

8. Verify that $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, where $y_1(t) = 2e^{4t} - e^{-2t}$ and $y_2(t) = 2e^{4t} + e^{-2t}$ is a solution of the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \boldsymbol{y}; \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

9. Verify that $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, where $y_1(t) = e^t + 2te^t$ and $y_2(t) = 4te^t$ is a solution of the initial value problem
$$\boldsymbol{y}' = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \boldsymbol{y}; \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

10. Verify that $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, where $y_1(t) = \cos 2t - 2\sin 2t$ and $y_2(t) = -\cos 2t$ is a solution of the initial value problem
$$\boldsymbol{y}' = \begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix} \boldsymbol{y}; \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Rewrite each of the following initial value problems for an ordinary differential equation as an initial value problem for a first order system of ordinary differential equations.

11. $y'' + 5y' + 6y = e^{2t}, \quad y(0) = 1,\ y'(0) = -2.$
Solution: Let $y_1 = y$ and $y_2 = y'$. Then $y_1' = y' = y_2$ and $y_2' = y'' = -5y' - 6y + e^{2t} = -6y_1 - 5y_2 + e^{2t}$. Letting $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this can be expressed in vector form (see Equation (6.1.7)) as
$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}; \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

12. $y'' + k^2 y = 0, \quad y(0) = -1,\ y'(0) = 0$

13. $y'' - k^2 y = 0, \quad y(0) = -1,\ y'(0) = 0$

14. $y'' + k^2 y = A\cos\omega t, \quad y(0) = 0,\ y'(0) = 0$

15. $ay'' + by' + cy = 0, \quad y(0) = \alpha,\ y'(0) = \beta$

16. $ay'' + by' + cy = A\sin\omega t, \quad y(0) = \alpha,\ y'(0) = \beta$

17. $t^2 y'' + 2ty' + y = 0, \quad y(1) = -2,\ y'(1) = 3$

6.2 Linear Systems of Differential Equations

This section and the next will be devoted to the theoretical underpinnings of linear systems of ordinary differential equations which accrue from the main theorem of existence and uniqueness of solutions of such systems. As with Picard's existence and uniqueness theorem (Theorem 1.5.2) for first order ordinary differential equations, and the similar theorem for linear second order equations (Theorem 3.1.6), we will not prove this theorem, but rather show how it leads to immediately useful information to assist us in knowing when we have found all solutions.

A first order system $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ in standard form is **linear** provided $\boldsymbol{f}(t, \boldsymbol{y}) = A(t)\boldsymbol{y} + \boldsymbol{q}(t)$ where $A(t) = [a_{ij}(t)]$ is an $n \times n$ matrix of functions, while

$$\boldsymbol{q}(t) = \begin{bmatrix} q_1(t) \\ \vdots \\ q_n(t) \end{bmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

are $n \times 1$ matrices. Thus the standard description of a first order linear system in matrix form is

$$\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{q}(t), \tag{1}$$

while, if the matrix equation is written out in terms of the unknown functions y_1, y_2, ..., y_n, then (1) becomes

$$\begin{array}{l} y_1' = a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + q_1(t) \\ y_2' = a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + q_2(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n' = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + q_n(t). \end{array} \tag{2}$$

For example, the matrix equation

$$\boldsymbol{y}' = \begin{bmatrix} 1 & -t \\ e^{-t} & -1 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} \cos t \\ 0 \end{bmatrix}$$

and the system of equations

$$\begin{array}{rcl} y_1' &=& y_1 - ty_2 + \cos t \\ y_2' &=& e^{-t}y_1 - y_2 \end{array}$$

have the same meaning.

It is convenient to state most of our results on linear systems of ordinary differential equations in the language of matrices and vectors. To this end the following terminology will be useful. A property $\mathbb{P}$ of functions will be said to be satisfied for a matrix $A(t) = [a_{ij}(t)]$ of functions if it is satisfied for *all* of the functions $a_{ij}(t)$ which make up the matrix. In particular:

1. $A(t)$ is **defined** on an interval I of $\mathbb{R}$ if each $a_{ij}(t)$ is defined on I.

2. $A(t)$ is **continuous** on an interval I of $\mathbb{R}$ if each $a_{ij}(t)$ is continuous on I. For instance, the matrix
$$A(t) = \begin{bmatrix} \dfrac{1}{t+2} & \cos 2t \\ e^{-2t} & \dfrac{1}{(2t-3)^2} \end{bmatrix}$$
is continuous on each of the intervals $I_1 = (-\infty, -2)$, $I_2 = (-2, 3/2)$ and $I_3 = (3/2, \infty)$, but it is not continuous on the interval $I_4 = (0, 2)$.

3. $A(t)$ is **differentiable** on an interval I of $\mathbb{R}$ if each $a_{ij}(t)$ is differentiable on I. Moreover, $A'(t) = [a'_{ij}(t)]$. That is, the matrix $A(t)$ is differentiated by differentiating each entry of the matrix. For instance, for the matrix $A(t)$ in the previous item,
$$A'(t) = \begin{bmatrix} \dfrac{-1}{(t+2)^2} & -2\sin 2t \\ -2e^{-2t} & \dfrac{-4}{(2t-3)^3} \end{bmatrix}.$$

4. $A(t)$ is **integrable** on an interval I of $\mathbb{R}$ if each $a_{ij}(t)$ is integrable on I. Moreover, the integral of $A(t)$ on the interval $[a, b]$ is computed by computing the integral of each entry of the matrix, i.e., $\int_a^b A(t)\,dt = \left[\int_a^b a_{ij}(t)\,dt\right]$. For the matrix $A(t)$ of item 2 above, this gives
$$\int_0^1 A(t)\,dt = \begin{bmatrix} \int_0^1 \dfrac{1}{t+2}\,dt & \int_0^1 \cos 2t\,dt \\ \int_0^1 e^{-2t}\,dt & \int_0^1 \dfrac{1}{(2t-3)^2}\,dt \end{bmatrix} = \begin{bmatrix} \ln\frac{3}{2} & \frac{1}{2}\sin 2 \\ \frac{1}{2}(1-e^{-2}) & \frac{1}{3} \end{bmatrix},$$
while, if $t \in I_2 = (-2, 3/2)$, then
$$\int_0^t A(u)\,du = \begin{bmatrix} \int_0^t \dfrac{1}{u+2}\,du & \int_0^t \cos 2u\,du \\ \int_0^t e^{-2u}\,du & \int_0^1 \dfrac{1}{(2u-3)^2}\,du \end{bmatrix} = \begin{bmatrix} \ln\dfrac{t+2}{2} & \frac{1}{2}\sin 2t \\ \frac{1}{2}(1-e^{-2t}) & \dfrac{-1}{2(2t-3)} - \frac{1}{6} \end{bmatrix}.$$

5. If each entry $a_{ij}(t)$ of $A(t)$ is of exponential type (see the definition on page 201), we can take the Laplace transform of $A(t)$, by taking the Laplace transform of each entry. That is $\mathcal{L}(A(t))(s) = [\mathcal{L}(a_{ij}(t))(s)]$. For example, if $A(t) = \begin{bmatrix} te^{-2t} & \cos 2t \\ e^{3t}\sin t & (2t-e)^2 \end{bmatrix}$, this gives
$$\mathcal{L}(A(t))(s) = \begin{bmatrix} \mathcal{L}\left(te^{-2t}\right)(s) & \mathcal{L}(\cos 2t)(s) \\ \mathcal{L}(e^{3t}\sin t)(s) & \mathcal{L}\left((2t-3)^2\right)(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{(s+2)^2} & \dfrac{2}{s^2+4} \\ \dfrac{1}{(s-3)^2+1} & \dfrac{8e^{\frac{3t}{2}}}{s^3} \end{bmatrix}.$$

If $A(t)$ and $\boldsymbol{q}(t)$ are continuous matrix functions on an interval I, then a solution to the linear differential equation

$$\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{q}(t)$$

on the subinterval $J \subseteq I$, is a continuous matrix function $\boldsymbol{y}(t)$ on J such that

$$\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t) + \boldsymbol{q}(t)$$

for all $t \in J$. If moreover, $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$, then $\boldsymbol{y}(t)$ is a solution of the **initial value problem**

$$\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{q}(t), \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0. \tag{3}$$

Example 6.2.1. Verify that $\boldsymbol{y}(t) = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}$ is a solution of the initial value problem (3) on the interval $(-\infty, \infty)$ where

$$A(t) = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix}, \quad \boldsymbol{q}(t) = \begin{bmatrix} 0 \\ 6e^{3t} \end{bmatrix}, \ t_0 = 0 \text{ and } \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

▶ **Solution.** All of the functions in the matrices $A(t)$, $\boldsymbol{y}(t)$, and $\boldsymbol{q}(t)$ are differentiable on the entire real line $(-\infty, \infty)$ and $\boldsymbol{y}(t_0) = \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \boldsymbol{y}_0$. Moreover,

$$(*) \qquad \boldsymbol{y}'(t) = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix}$$

and

$$(**) \qquad A(t)\boldsymbol{y}(t) + \boldsymbol{q}(t) = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} + \begin{bmatrix} 0 \\ 6e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix}.$$

Since $(*)$ and $(**)$ agree, $\boldsymbol{y}(t)$ is a solution of the initial value problem. ◀

The Existence and Uniqueness Theorem

The following result is the fundamental foundational result of the current theory. It is the result which guarantees that if we can find a solution of a linear initial value problem by any means whatsoever, then we know that we have found the only possible solution.

Theorem 6.2.2 (Existence and Uniqueness). [1] *Suppose that the $n \times n$ matrix function $A(t)$ and the $n \times 1$ matrix function $\boldsymbol{q}(t)$ are both continuous on an interval I in $\mathbb{R}$. Let $t_0 \in I$. Then for every choice of the vector $\boldsymbol{y}_0$, the initial value problem*

$$\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{q}(t), \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

has a unique solution $\boldsymbol{y}(t)$ which is defined on the same interval I.

Remark 6.2.3. How is this theorem related to existence and uniqueness theorems we have stated previously?

- If $n = 1$ then this theorem is just Corollary 1.3.9. In this case we have actually proved the result by exhibiting a formula for the unique solution. We are not so lucky for general n. There is no formula like Equation (7) which is valid for the solutions of linear initial value problems if $n > 1$.

- Theorem 3.1.6 is a corollary of Theorem 6.2.2. Indeed, if $n = 2$,

$$A(t) = \begin{bmatrix} 0 & 1 \\ -b(t) & -a(t) \end{bmatrix},$$

$\boldsymbol{q}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$, $\boldsymbol{y}_0 = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$, and $\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$, then the second order linear initial value problem

$$y'' + a(t)y' + b(t)y = f(t), \quad y(t_0) = y_0, \ y'(t_0) = y_1$$

has the solution $y(t)$ if and only if the first order linear system

$$\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{q}(t), \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

has the solution $\boldsymbol{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. You should convince yourself of the validity of this statement.

Example 6.2.4. Let $n = 2$ and consider the initial value problem (3) where

$$A(t) = \begin{bmatrix} -t & \dfrac{1}{t+1} \\ \dfrac{1}{t^2-2} & t^2 \end{bmatrix}, \quad \boldsymbol{q}(t) = \begin{bmatrix} e^{-t} \\ \cos t \end{bmatrix}, \quad t_0 = 0, \quad \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Determine the largest interval I on which a solution to (3) is guaranteed by Theorem 6.2.2.

[1] A Proof of this result can be found in the text *An Introduction to Ordinary Differential Equations* by Earl Coddington, Prentice-Hall, (1961), Page 256.

▶ **Solution.** All of the entries in all the matrices above are continuous on the entire real line except that $\dfrac{1}{t+1}$ is not continuous for $t = -1$ and $\dfrac{1}{t^2-2}$ is not continuous for $t = \pm\sqrt{2}$. Thus the largest interval I containing 0 for which all of the matrix entries are continuous is $I = (-1, \sqrt{2})$. The theorem applies on this interval, and on no larger interval containing 0. ◀

For first order differential equations, the Picard approximation algorithm (Algorithm 1.5.1) provides an algorithmic procedure for finding an approximate solution to a first order initial value problem $y' = f(t, y)$, $y'(t_0) = y_0$. For first order systems, the Picard approximation algorithm also works. We will state the algorithm only for linear first order systems and then apply it to *constant coefficient* first order systems, where we will be able to see an immediate analogy to the simple linear equation $y' = ay$, $y'(0) = c$, which, as we know from Chapter 1, has the solution $y(t) = ce^{at}$.

Algorithm 6.2.5 (Picard Approximation for Linear Systems). Perform the following sequence of steps to produce an *approximate solution* to the initial value problem (3).

(i) A rough initial approximation to a solution is given by the constant function

$$\boldsymbol{y}_0(t) := \boldsymbol{y}_0.$$

(ii) Insert this initial approximation into the right hand side of Equation (3) and obtain the first approximation

$$\boldsymbol{y}_1(t) := \boldsymbol{y}_0 + \int_{t_0}^{t} (A(u)\boldsymbol{y}_0(u) + \boldsymbol{q}(u))\, du.$$

(iii) The next step is to generate the second approximation in the same way; i.e.,

$$\boldsymbol{y}_2(t) := \boldsymbol{y}_0 + \int_{t_0}^{t} (A(u)\boldsymbol{y}_1(u) + \boldsymbol{q}(u))\, du.$$

(iv) At the n-th stage of the process we have

$$\boldsymbol{y}_n(t) := \boldsymbol{y}_0 + \int_{t_0}^{t} (A(u)\boldsymbol{y}_{n-1}(u) + \boldsymbol{q}(u))\, du,$$

which is defined by substituting the previous approximation $\boldsymbol{y}_{n-1}(t)$ into the right hand side of Equation (3).

As in the case of first order equations, under the hypotheses of the Existence and Uniqueness for linear systems, the sequence of vector functions $\boldsymbol{y}_n(t)$ produced by the Picard Approximation algorithm will converge on the interval I to the unique solution of the initial value problem (3).

Example 6.2.6. We will consider the Picard Approximation algorithm in the special case where the coefficient matrix $A(t)$ is *constant*, so that we can write $A(t) = A$, the function $\boldsymbol{q}(t) = \mathbf{0}$, and the initial point $t_0 = 0$. In this case the initial value problem (3) becomes

$$\boldsymbol{y}' = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{4}$$

and we get the following sequence of Picard approximations $\boldsymbol{y}_n(t)$ to the solution $\boldsymbol{y}(t)$ of (4).

$$\begin{aligned}
\boldsymbol{y}_0(t) &= \boldsymbol{y}_0 \\
\boldsymbol{y}_1(t) &= \boldsymbol{y}_0 + \int_0^t A\boldsymbol{y}_0\, du \\
&= \boldsymbol{y}_0 + A\boldsymbol{y}_0 t \\
\boldsymbol{y}_2(t) &= \boldsymbol{y}_0 + \int_0^t A\boldsymbol{y}_1(u)\, du \\
&= \boldsymbol{y}_0 + \int_0^t A(\boldsymbol{y}_0 + A\boldsymbol{y}_0 u)\, du \\
&= \boldsymbol{y}_0 + A\boldsymbol{y}_0 t + \frac{1}{2}A^2\boldsymbol{y}_0 t^2 \\
\boldsymbol{y}_3(t) &= \int_0^t A\left(\boldsymbol{y}_0 + A\boldsymbol{y}_0 u + \frac{1}{2}A^2\boldsymbol{y}_0 u^2\right) du \\
&= \boldsymbol{y}_0 + A\boldsymbol{y}_0 t + \frac{1}{2}A^2\boldsymbol{y}_0 t^2 + \frac{1}{6}A^3\boldsymbol{y}_0 t^3 \\
&\vdots \\
\boldsymbol{y}_n(t) &= \boldsymbol{y}_0 + A\boldsymbol{y}_0 t + \frac{1}{2}A^2\boldsymbol{y}_0 t^2 + \cdots + \frac{1}{n!}A^n\boldsymbol{y}_0 t^n.
\end{aligned}$$

Notice that we may factor a $\boldsymbol{y}_0$ out of each term on the right hand side of $\boldsymbol{y}_n(t)$. This gives the following expression for the function $\boldsymbol{y}_n(t)$:

$$\boldsymbol{y}_n(t) = \left(I_n + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n\right)\boldsymbol{y}_0 \tag{5}$$

where I_n denotes the identity matrix of size n. If you recall the Taylor series expansion for the exponential function e^{at}:

$$e^{at} = 1 + at + \frac{1}{2}(at)^2 + \frac{1}{3!}(at)^3 + \cdots + \frac{1}{n!}(at)^n + \cdots$$

you should immediately see a similarity. If we replace the scalar a with the $n \times n$ matrix A and the scalar 1 with the identity matrix I_n then we can *define* e^{At} to be the sum of the resulting series. That is,

$$e^{At} = I_n + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 + \cdots + \frac{1}{n!}(At)^n + \cdots . \tag{6}$$

It is not difficult (but we will not do it) to show that the series we have written down for defining e^{At} in fact converges for any $n \times n$ matrix A, and the resulting sum is an $n \times n$ matrix of functions of t. That is

$$e^{At} = \begin{bmatrix} h_{11}(t) & h_{12}(t) & \cdots & h_{1n}(t) \\ h_{21}(t) & h_{22}(t) & \cdots & h_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1}(t) & h_{n2}(t) & \cdots & h_{nn}(t) \end{bmatrix}.$$

It is *not*, however, obvious what the functions $h_{ij}(t)$ are. Much of the remainder of this chapter will be concerned with precisely that problem. For now, we simply want to observe that the functions $\boldsymbol{y}_n(t)$ (see Equation (5)) computed from the Picard approximation algorithm converge to $e^{At}\boldsymbol{y}_0$, that is the matrix function e^{At} multiplied by the constant vector $\boldsymbol{y}_0$ from the initial value problem (4). Hence we have arrived at the following fact: The unique solution to (4) is

$$\boxed{\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0.} \tag{7}$$

Following are a few examples where we can compute the matrix exponential e^{At} with only the definition.

Example 6.2.7. Compute e^{At} for each of the following constant matrices A.

1. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$. (In general 0_k denotes the $k \times k$ matrix, all of whose entries are 0.)

   ► **Solution.** In this case $A^n t^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all n. Hence,

$$\begin{aligned} e^{0_2 t} = e^{At} &= I_2 + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \\ &= I_2 + 0_2 + 0_2 + \cdots \\ &= I_2. \end{aligned}$$

   Similarly, $e^{0_n t} = I_n$. This is the matrix analog of the fact $e^0 = 1$. ◄

2. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

▶ **Solution.** In this case the powers of the matrix A are easy to compute. In fact

$$A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix}, \quad \cdots \quad A^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix},$$

so that

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 3t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4t^2 & 0 \\ 0 & 9t^2 \end{bmatrix} + \\ &\qquad + \frac{1}{3!}\begin{bmatrix} 8t^3 & 0 \\ 0 & 27t^3 \end{bmatrix} + \cdots + \frac{1}{n!}\begin{bmatrix} 2^nt^n & 0 \\ 0 & 3^nt^n \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 + 2t + \frac{1}{2}4t^2 + \cdots + \frac{1}{n!}2^nt^n + \cdots & 0 \\ 0 & 1 + 3t + \frac{1}{2}9t^2 + \cdots + \frac{1}{n!}3^nt^n + \cdots \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}. \end{aligned}$$

◀

3. $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

▶ **Solution.** There is clearly nothing special about the numbers 2 and 3 on the diagonal of the matrix in the last example. The same calculation shows that

$$e^{At} = e^{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} t} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}. \tag{8}$$

◀

4. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

▶ **Solution.** In this case, check that $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$. Then $A^n = 0_2$ for all $n \geq 2$. Hence,

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \\ &= I + At \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Note that in this case, the individual entries of e^{At} do not look like exponential functions e^{at} at all. ◀

5. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

▶ **Solution.** We leave it as an exercise to compute the powers of the matrix A. You should find $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$, $A^5 = A$, $A^6 = A^2$, etc. That is, the powers repeat with period 4. Then

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & t^3 \\ -t^3 & 0 \end{bmatrix} + \frac{1}{4!}\begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 + \cdots & -t + \frac{1}{3!}t^3 - \frac{1}{5!}t^5 + \cdots \\ t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots & 1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 + \cdots \end{bmatrix} \\ &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}. \end{aligned}$$

In this example also the individual entries of e^{At} are not themselves exponential functions. ◀

Example 6.2.8. Use Equation (7) and the calculation of e^{At} from the corresponding item in the previous example to solve the initial value problem

$$\boldsymbol{y}' = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

for each of the following matrices A.

1. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$.

▶ **Solution.** By Equation (7), the solution $\boldsymbol{y}(t)$ is given by

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 = I_2\boldsymbol{y}_0 = \boldsymbol{y}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{9}$$

That is, the solution of the vector differential equation $\boldsymbol{y}' = \mathbf{0}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is the constant function $\boldsymbol{y}(t) = \boldsymbol{y}_0$. In terms of the component functions $y_1(t)$, $y_2(t)$, the system of equations we are considering is

$$\begin{aligned} y_1' &= 0, \quad y_1(0) = c_1 \\ y_2' &= 0, \quad y_2(0) = c_2 \end{aligned}$$

and this clearly has the solution $y_1(t) = c_1$, $y_2(t) = c_2$, which agrees with (9). ◀

2. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

▶ **Solution.** Since in this case, $e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$, the solution of the initial value problem is

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1e^{2t} \\ c_2e^{3t} \end{bmatrix}.$$

Again, in terms of the component functions $y_1(t)$, $y_2(t)$, the system of equations we are considering is

$$\begin{aligned} y_1' &= 2y_1, \quad y_1(0) = c_1 \\ y_2' &= 3y_2, \quad y_2(0) = c_2. \end{aligned}$$

Since the first equation does not involve y_2 and the second equation does not involve y_1, what we really have is two *independent* first order linear equations. The first equation clearly has the solution $y_1(t) = c_1e^{2t}$ and the second clearly has the solution $y_2(t) = c_2e^{3t}$, which agrees with the vector description provided by Equation (7). (If the use of the word clearly is *not* clear, then you are advised to review Section 1.3.) ◀

3. $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

▶ **Solution.** Since in this case, $e^{At} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$, the solution of the initial value problem is

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1e^{at} \\ c_2e^{bt} \end{bmatrix}.$$

◀

4. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

▶ **Solution.** In this case $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, so the solution of the initial value problem is

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 + tc_2 \\ c_2 \end{bmatrix}.$$

Again, for comparative purposes, we will write this equation as a system of two equations in two unknowns:

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = c_1 \\ y_2' &= 0, \quad y_2(0) = c_2. \end{aligned}$$

In this case also, it is easy to see directly what the solution of the system is and to see that it agrees with that computed by Equation (7). Indeed, the second equation says that $y_2(t) = c_2$, and then the first equation implies that $y_1(t) = c_1 + tc_2$ by integration.

◀

5. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

▶ **Solution.** The solution of the initial value problem is

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \cos t - c_2 \sin t \\ c_1 \sin t + c_2 \cos t \end{bmatrix}.$$

◀

Exercises

Compute the derivative of each of the following matrix functions.

1. $A(t) = \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$

2. $A(t) = \begin{bmatrix} e^{-3t} & t \\ t^2 & e^{2t} \end{bmatrix}$

3. $A(t) = \begin{bmatrix} e^{-t} & te^{-t} & t^2e^{-t} \\ 0 & e^{-t} & te^{-t} \\ 0 & 0 & e^{-t} \end{bmatrix}$

4. $\boldsymbol{y}(t) = \begin{bmatrix} t \\ t^2 \\ \ln t \end{bmatrix}$

5. $A(t) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

6. $\boldsymbol{v}(t) = \begin{bmatrix} e^{-2t} & \ln(t^2+1) & \cos 3t \end{bmatrix}$

For each of the following matrix functions, compute the requested integral.

7. Compute $\int_0^\pi A(t)\,dt$ if $A(t) = \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$.

8. Compute $\int_0^1 A(t)\,dt$ if $A(t) = \frac{1}{2}\begin{bmatrix} e^{2t}+e^{-2t} & e^{2t}-e^{-2t} \\ e^{-2t}-e^{2t} & e^{2t}+e^{-2t} \end{bmatrix}$

9. Compute $\int_1^2 \boldsymbol{y}(t)\,dt$ for the matrix $\boldsymbol{y}(t)$ of Exercise 4.

10. Compute $\int_1^5 A(t)\,dt$ for the matrix $A(t)$ of Exercise 5.

11. On which of the following intervals is the matrix function $A(t) = \begin{bmatrix} t & (t+1)^{-1} \\ (t-1)^{-2} & t+6 \end{bmatrix}$ continuous?
 (a) $I_1 = (-1, 1)$ (b) $I_2 = (0, \infty)$ (c) $I_3 = (-1, \infty)$
 (d) $I_4 = (-\infty, -1)$ (e) $I_5 = (2, 6)$

If $A(t) = [a_{ij}(t)]$ is a matrix of functions, then the Laplace transform of $A(t)$ can be defined by taking the Laplace transform of each function $a_{ij}(t)$. That is,

$$\mathcal{L}(A(t))(s) = [\mathcal{L}(a_{ij}(t))(s)].$$

For example, if $A(t) = \begin{bmatrix} e^{2t} & \sin 2t \\ e^{2t}\cos 3t & t \end{bmatrix}$, then

$$\mathcal{L}(A(t))(s) = \begin{bmatrix} \frac{1}{s-2} & \frac{2}{s^2+4} \\ \frac{s-2}{(s-2)^2+9} & \frac{1}{s^2} \end{bmatrix}.$$

Compute the Laplace transform of each of the following matrix functions.

12. $A(t) = \begin{bmatrix} 1 & t \\ t^2 & e^{2t} \end{bmatrix}$

13. $A(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

14. $A(t) = \begin{bmatrix} t^3 & t\sin t & te^{-t} \\ t^2 - t & e^{3t}\cos 2t & 3 \end{bmatrix}$

15. $A(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$

16. $A(t) = e^t \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + e^{-t} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

17. $A(t) = \begin{bmatrix} 1 & \sin t & 1 - \cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}$

The inverse Laplace transform of a matrix function is also defined by taking the inverse Laplace transform of each entry of the matrix. For example,

$$\mathcal{L}^{-1}\left(\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ \frac{1}{s^3} & \frac{1}{s^4} \end{bmatrix}\right) = \begin{bmatrix} 1 & t \\ \frac{t^2}{2} & \frac{t^4}{6} \end{bmatrix}.$$

Compute the inverse Laplace transform of each matrix function:

18. $\begin{bmatrix} \frac{1}{s} & \frac{2}{s^2} & \frac{6}{s^3} \end{bmatrix}$

19. $\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ \frac{s}{s^2 - 1} & \frac{s}{s^2 + 1} \end{bmatrix}$

20. $\begin{bmatrix} \frac{1}{s-1} & \frac{1}{s^2 - 2s + 1} \\ \frac{4}{s^3 + 2s^2 - 3s} & \frac{1}{s^2 + 1} \\ \frac{3s}{s^2 + 9} & \frac{1}{s - 3} \end{bmatrix}$

21. $\begin{bmatrix} \frac{2s}{s^2-1} & \frac{2}{s^2-1} \\ \frac{2}{s^2-1} & \frac{2s}{s^2-1} \end{bmatrix}$

For each matrix A given below:

(i) Compute $(sI - A)^{-1}$.

(ii) Compute the inverse Laplace transform of $(sI - A)^{-1}$.

22. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

25. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

26. Let $A(t) = \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}$ and consider the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Use Picard's method to calculate the first four terms, $\mathbf{y_0}, \cdots, \mathbf{y_3}$.

(b) Make a conjecture about what the n-th term will be. Do you recognize the series?

(c) Verify that $\mathbf{y}(t) = \begin{bmatrix} e^{t^2/2} \\ e^{t^2/2} \end{bmatrix}$ is a solution. Are there any other solutions possible? Why or Why not?

27. Let $A(t) = \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}$ and consider the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(a) Use Picard's method to calculate the first four terms, $\mathbf{y_0}, \cdots, \mathbf{y_3}$.

(b) Verify that $\mathbf{y}(t) = \begin{bmatrix} \cos t^2/2 \\ -\sin t^2/2 \end{bmatrix}$ is a solution. Are there any other solutions possible? Why or Why not?

28. Let $A(t) = \begin{bmatrix} t & t \\ -t & -t \end{bmatrix}$ and consider the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Use Picard's method to calculate the first four terms, $\mathbf{y_0}, \cdots, \mathbf{y_3}$.

(b) Deduce the solution.

29. Verify the product rule for matrix functions. That is, if $A(t)$ and $B(t)$ are matrix functions which can be multiplied and $C(t) = A(t)B(t)$ is the product, then

$$C'(t) = A'(t)B(t) + A(t)B'(t).$$

Hint: Write the ij term of $C(t)$ as $c_{ij}(t) = \sum_{k=1}^{r} a_{ik}(t)b_{kj}(t)$ (where r is the number of columns of $A(t)$ = the number of rows of $B(t)$), and use the ordinary product and sum rules for derivatives.

What is the largest interval containing 0 on which the initial value problem

$$\boldsymbol{y}' = A(t)\boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

is guaranteed by Theorem 4.2.2 to have a solution, assuming:

30. $A(t) = \begin{bmatrix} 0 & 1 \\ (t^2+2)^{-1} & \cos t \end{bmatrix}$

31. $A(t) = \begin{bmatrix} (t+4)^{-2} & t^2+4 \\ \ln(t-3) & (t+2)^{-4} \end{bmatrix}$

32. $A(t) = \begin{bmatrix} \frac{t+2}{t^2-5t+6} & t \\ t^2 & t^3 \end{bmatrix}$

33. $A(t) = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$

34. Let $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) Show that

$$N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Using the above calculations, compute e^{Nt}.

(c) Solve the initial value problem

$$\boldsymbol{y}' = N\boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(d) Compute the Laplace transform of e^{Nt}, which you calculated in Part (b).

(e) Compute the matrix $(sI - N)^{-1}$. Do you see a similarity to the matrix computed in the previous part?

35. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Verify that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all natural numbers n.

(b) Using part (a), verify, directly from the definition, that

$$e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$

(c) Now solve the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ for each of the following initial conditions $\boldsymbol{y}_0$.

(i) $\boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, (ii) $\boldsymbol{y}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, (iii) $\boldsymbol{y}_0 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, (iv) $\boldsymbol{y}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

(d) Compute the Laplace transform of e^{At}.

(e) Compute $(sI - A)^{-1}$ and compare to the matrix computed in Part (d).

36. One of the fundamental properties of the exponential function is the formula $e^{a+b} = e^a e^b$. The goal of this exercise is to show, by means of a concrete example, that the analog of this fundamental formula is not true for the matrix exponential function (at least without some additional assumptions). From the calculations of Example 4.2.7, you know that if $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$$e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \quad \text{and} \quad e^{Bt} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

(a) Show that $e^{At}e^{Bt} = \begin{bmatrix} e^{2t} & te^{3t} \\ 0 & e^{3t} \end{bmatrix}$.

(b) Let $\boldsymbol{y}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and let $\boldsymbol{y}(t) = e^{At}e^{Bt}\boldsymbol{y}_0$. Compute $\boldsymbol{y}'(t)$ and $(A+B)\boldsymbol{y}(t)$. Are these two functions the same?

(c) What do these calculations tell you about the possible equality of the matrix functions $e^{At}e^{Bt}$ and $e^{(A+B)t}$ for these particular A and B?

(d) We will see later that the formula $e^{(A+B)t} = e^{At}e^{Bt}$ is valid *provided* that $AB = BA$. Check that $AB \neq BA$ for the matrices of this exercise.

6.3 Linear Homogeneous Equations

This section will be concerned with using the fundamental existence and uniqueness theorem for linear systems (Theorem 6.2.2) to describe the solution set for a linear homogeneous system of ordinary differential equations

$$\boldsymbol{y}' = A(t)\boldsymbol{y}. \tag{1}$$

The main result will be similar to the description given by Theorem 3.2.4 for linear homogeneous second order equations.

Recall that if $A(t)$ is a continuous $n \times n$ matrix function on an interval I, then a **solution** to system (1) is an $n \times 1$ matrix function $\boldsymbol{y}(t)$ such that $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t)$ for all $t \in I$. Since this is equivalent to the statement

$$\boldsymbol{y}'(t) - A(t)\boldsymbol{y}(t) = \boldsymbol{0} \text{ for all } t \in I,$$

to be consistent with the language of solution sets used in Chapter 3, we will denote the set of all solutions of (1) by $\mathcal{S}^0_{\boldsymbol{L}}$, where $\boldsymbol{L} = \boldsymbol{D} - A(t)$ is the (vector) differential operator which acts on the vector function $\boldsymbol{y}(t)$ by the rule

$$\boldsymbol{L}(\boldsymbol{y}(t)) = (\boldsymbol{D} - A(t))(\boldsymbol{y}(t)) = \boldsymbol{y}'(t) - A(t)\boldsymbol{y}(t).$$

Thus

$$\mathcal{S}^0_{\boldsymbol{L}} = \{\boldsymbol{y}(t) : \boldsymbol{L}(\boldsymbol{y}(t)) = \boldsymbol{0}\}.$$

Let $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ be two solutions of system (1), i.e., $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ are in $\mathcal{S}_{\boldsymbol{L}}^0$ and let $\boldsymbol{y}(t) = c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t)$ where c_1 and c_2 are scalars (either real or complex). Then

$$\begin{aligned}
\boldsymbol{y}'(t) &= (c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t))' \\
&= c_1\boldsymbol{y}_1'(t) + c_2\boldsymbol{y}_2'(t) \\
&= c_1A(t)\boldsymbol{y}_1(t) + c_2A(t)\boldsymbol{y}_2(t) \\
&= A(t)\left(c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t)\right) \\
&= A(t)\boldsymbol{y}(t).
\end{aligned}$$

Thus every linear combination of two solutions of (1) is again a solution. which in the language of linear algebra means that $\mathcal{S}_{\boldsymbol{L}}^0$ is a **vector space**. We say that a set of vectors

$$\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$$

in a vector space $\boldsymbol{V}$ is a **basis** of $\boldsymbol{V}$ if the set $\mathcal{B}$ is linearly independent and if every vector $\boldsymbol{v}$ in $\boldsymbol{V}$ can be written as a linear combination

$$\boldsymbol{v} = \lambda_1\boldsymbol{v}_1 + \cdots + \lambda_k\boldsymbol{v}_k.$$

The number k of vectors in a basis $\mathcal{B}$ of $\boldsymbol{V}$ is known as the **dimension** of $\boldsymbol{V}$. Thus $\mathbb{R}^2$ has dimension 2 since it has a basis $\boldsymbol{e}_1 = (1, 0)$, $\boldsymbol{e}_2 = (0, 1)$ consisting of 2 vectors, $\mathbb{R}^3$ has dimension 3 since it has a basis $\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{e}_2 = (0, 1, 0)$, $\boldsymbol{e}_3 = (0, 0, 1)$ consisting of 3 vectors, etc. The main theorem on solutions of linear homogeneous systems can be expressed most conveniently in the language of vector spaces.

Theorem 6.3.1. *If the $n \times n$ matrix $A(t)$ is continuous on an interval I, then the solution set $\mathcal{S}_{\boldsymbol{L}}^0$ of the homogeneous system*

$$\boldsymbol{y}' = A(t)\boldsymbol{y} \tag{2}$$

is a vector space of dimension n. In other words,

1. *There are n linearly independent solutions of (2) in $\mathcal{S}_{\boldsymbol{L}}^0$.*

2. *If $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_n \in \mathcal{S}_{\boldsymbol{L}}^0$ are independent solutions of (2), and $\boldsymbol{\varphi}$ is any function in $\mathcal{S}_{\boldsymbol{L}}^0$, then φ can be written as*

$$\varphi = c_1\boldsymbol{\varphi}_1 + \cdots + c_n\boldsymbol{\varphi}_n$$

for some scalars $c_1, \ldots, c_n \in \mathbb{R}$.

Proof. To keep the notation as explicit as possible, we will only present the proof in the case $n = 2$. You should compare this proof with that of Theorem 3.2.4. To start with, let

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and let $t_0 \in I$. By Theorem 6.2.2 there are vector functions $\boldsymbol{\psi}_1(t)$ and $\boldsymbol{\psi}_2(t)$ defined for all $t \in I$ and which satisfy the initial conditions

$$\boldsymbol{\psi}_i(t_0) = \boldsymbol{e}_i \text{ for } i = 1,\ 2. \tag{3}$$

Suppose there is a dependence relation $c_1\boldsymbol{\psi}_1 + c_2\boldsymbol{\psi}_2 = \mathbf{0}$. This means that

$$c_1\boldsymbol{\psi}_1(t) + c_2\boldsymbol{\psi}_2(t) = \mathbf{0}$$

for all $t \in I$. Applying this equation to the particular point t_0 gives

$$\mathbf{0} = c_1\boldsymbol{\psi}_1(t_0) + c_2\boldsymbol{\psi}_2(t_0) = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Thus $c_1 = 0$ and $c_2 = 0$ so that $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are linearly independent. This proves (1).

Now suppose that $\boldsymbol{\varphi} \in \mathcal{S}^0_{\boldsymbol{L}}$. Evaluating at t_0 gives

$$\boldsymbol{\varphi}(t_0) = \begin{bmatrix} r \\ s \end{bmatrix}.$$

Now define $\boldsymbol{\psi} \in \mathcal{S}^0_{\boldsymbol{L}}$ by $\boldsymbol{\psi} = r\boldsymbol{\psi}_1 + s\boldsymbol{\psi}_2$. Note that $\boldsymbol{\psi} \in \mathcal{S}^0_{\boldsymbol{L}}$ since $\mathcal{S}^0_{\boldsymbol{L}}$ is a vector space. Moreover,

$$\boldsymbol{\psi}(t_0) = r\boldsymbol{\psi}_1(t_0) + s\boldsymbol{\psi}_2(t_0) = r\boldsymbol{e}_1 + s\boldsymbol{e}_2 = \begin{bmatrix} r \\ s \end{bmatrix} = \boldsymbol{\varphi}(t_0).$$

This means that $\boldsymbol{\varphi}$ and $\boldsymbol{\psi} = r\boldsymbol{\psi}_1 + s\boldsymbol{\psi}_2$ are two elements of $\mathcal{S}^0_{\boldsymbol{L}}$ which have the same value at t_0. By the uniqueness part of Theorem 6.2.2, they are equal.

Now suppose that $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are any two linearly independent solutions of (2) in $\mathcal{S}^0_{\boldsymbol{L}}$. From the argument of the previous paragraph, there are scalars a, b, c, d so that

$$\begin{aligned} \boldsymbol{\varphi}_1 &= a\boldsymbol{\psi}_1 + b\boldsymbol{\psi}_2 \\ \boldsymbol{\varphi}_2 &= c\boldsymbol{\psi}_1 + d\boldsymbol{\psi}_2 \end{aligned}$$

which in matrix form can be written

$$\begin{bmatrix} \boldsymbol{\varphi}_1 & \boldsymbol{\varphi}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

We multiply both sides of this matrix equation on the right by the adjoint $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ to obtain

$$\begin{bmatrix} \boldsymbol{\varphi}_1 & \boldsymbol{\varphi}_2 \end{bmatrix} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 \end{bmatrix} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 \end{bmatrix} (ad-bc).$$

Suppose $ad - bc = 0$. Then

$$\begin{aligned} d\boldsymbol{\varphi}_1 - b\boldsymbol{\varphi}_2 &= 0 \\ \text{and} \quad -c\boldsymbol{\varphi}_1 + a\boldsymbol{\varphi}_2 &= 0. \end{aligned}$$

But since $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are independent this implies that a, b, c, and d are zero which in turn implies that $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are both zero. But this cannot be. We conclude that $ad - bc \neq 0$. We can now write $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ each as a linear combination of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$. Specifically,

$$\begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} \boldsymbol{\varphi}_1 & \boldsymbol{\varphi}_2 \end{bmatrix} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Since $\boldsymbol{\varphi}$ is a linear combination of $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ it follows that $\boldsymbol{\varphi}$ is a linear combination of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$. □

The matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ that appears in the above proof is a useful theoretical criterion for determining if a pair of solutions $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$ in $\mathcal{S}^0_{\boldsymbol{L}}$ is linearly indpependent and hence a basis in the case $n = 2$. The proof shows:

1. $\begin{bmatrix} \boldsymbol{\varphi}_1(t_0) & \boldsymbol{\varphi}_2(t_0) \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

2. If the solutions $\boldsymbol{\varphi}_1(t)$, $\boldsymbol{\varphi}_2(t)$ are a basis of $\mathcal{S}^0_{\boldsymbol{L}}$, then $ad - bc = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \neq 0$, and moreover, this is true for *any* $t_0 \in I$.

3. The converse of the above statement is also true (and easy). Namely, if

$$\det \begin{bmatrix} \boldsymbol{\varphi}_1(t_0) & \boldsymbol{\varphi}_2(t_0) \end{bmatrix} \neq 0$$

for *some* $t_0 \in I$, then $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$ in $\mathcal{S}^0_{\boldsymbol{L}}$ is a basis of the solution space (always assuming $n = 2$).

Now assume that $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are any two solutions in $\mathcal{S}^0_{\boldsymbol{L}}$. Then we can form a 2×2 matrix of functions

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} \boldsymbol{\varphi}_1(t) & \boldsymbol{\varphi}_2(t) \end{bmatrix}$$

where each column is a solution to $\boldsymbol{y}' = A(t)\boldsymbol{y}$. We will say that $\boldsymbol{\Phi}(t)$ is a **fundamental matrix** for $\boldsymbol{y}' = A(t)\boldsymbol{y}$ if the columns are linearly independent, and hence form a basis of $\mathcal{S}^0_{\boldsymbol{L}}$. Then the above discussion is summarized in the following result.

Theorem 6.3.2. *If $A(t)$ is a 2×2 matrix of continuous functions on I, and if $\boldsymbol{\Phi}(t) = \begin{bmatrix}\boldsymbol{\varphi}_1(t) & \boldsymbol{\varphi}_2(t)\end{bmatrix}$ where each column is a solution to $\boldsymbol{y}' = A(t)\boldsymbol{y}$, then $\boldsymbol{\Phi}(t)$ is a fundamental matrix for $\boldsymbol{y}' = A(t)\boldsymbol{y}$ if and only if $\det \boldsymbol{\Phi}(t) \neq 0$ for at least one $t \in I$. If this is true for one $t \in I$, it is in fact true for all $t \in I$.*

Remark 6.3.3. The above theorem is also true, although we will not prove it, for $n \times n$ matrix systems $\boldsymbol{y}' = A(t)\boldsymbol{y}$, where a solution matrix consists of an $n \times n$ matrix

$$\boldsymbol{\Phi}(t) = \begin{bmatrix}\boldsymbol{\varphi}_1(t) & \cdots & \boldsymbol{\varphi}_n(t)\end{bmatrix}$$

where each column $\boldsymbol{\varphi}_i(t)$ is a solution to $\boldsymbol{y}' = A(t)\boldsymbol{y}$. Then $\boldsymbol{\Phi}(t)$ is a fundamental matrix, that is the columns are a basis for $\mathcal{S}^0_{\boldsymbol{L}}$ if and only if $\det \Phi(t) \neq 0$ for at least one $t \in I$.

Note that if a matrix B is written in columns, say

$$B = \begin{bmatrix}b_1 & \cdots & b_n\end{bmatrix},$$

then the matrix multiplication AB, if it is defined (which means the number of columns of A is the number of rows of B), can be written as

$$AB = A\begin{bmatrix}b_1 & \cdots & b_n\end{bmatrix} = \begin{bmatrix}Ab_1 & \cdots & Ab_n\end{bmatrix}.$$

In other words, multiply A by each column of B separately. For example, if

$$A = \begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -1\end{bmatrix},$$

then

$$\begin{aligned} AB &= \begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -1\end{bmatrix} \\ &= \begin{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} & \begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} & \begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}\end{bmatrix} \\ &= \begin{bmatrix}1 & -1 & 3\\ 2 & 3 & 1\end{bmatrix}. \end{aligned}$$

Now suppose that $\boldsymbol{\Phi}(t) = \begin{bmatrix}\boldsymbol{\varphi}_1(t) & \cdots & \boldsymbol{\varphi}_n(t)\end{bmatrix}$ is a fundamental matrix of solutions for $\boldsymbol{y}' = A(t)\boldsymbol{y}$. Then

$$\begin{aligned} \boldsymbol{\Phi}'(t) &= \begin{bmatrix}\boldsymbol{\varphi}_1'(t) & \cdots & \boldsymbol{\varphi}_n'(t)\end{bmatrix} \\ &= \begin{bmatrix}A(t)\boldsymbol{\varphi}_1(t) & \cdots & \boldsymbol{A}(t)\varphi_n(t)\end{bmatrix} \\ &= A(t)\begin{bmatrix}\boldsymbol{\varphi}_1(t) & \cdots & \boldsymbol{\varphi}_n(t)\end{bmatrix} \\ &= A(t)\boldsymbol{\Phi}(t). \end{aligned}$$

Thus the $n \times n$ matrix $\mathbf{\Phi}(t)$ satisfies the same differential equation, namely $\boldsymbol{y}' = A(t)\boldsymbol{y}$, as each of its columns. We summarize this discussion in the following theorem.

Theorem 6.3.4. *If $A(t)$ is a continuous $n \times n$ matrix of functions on an interval I, then an $n \times n$ matrix of functions $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous linear equation $\boldsymbol{y}' = A(t)\boldsymbol{y}$ if and only if*

$$\begin{aligned} \mathbf{\Phi}'(t) &= A(t)\Phi(t) \\ \text{and} \qquad \det \mathbf{\Phi}(t) &\neq 0. \end{aligned} \tag{4}$$

The second condition need only be checked for one value of $t \in I$.

Example 6.3.5. Show that

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix}$$

is a fundamental matrix for the system $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{y}$.

▶ **Solution.** First check that $\mathbf{\Phi}(t)$ is a solution matrix, i.e., check that the first condition of Equation (4) is satisfied. To see this, we calculate

$$\mathbf{\Phi}'(t) = \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} & -e^{-t} \\ 4e^{2t} & e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{\Phi}(t) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} & -e^{-t} \\ 4e^{2t} & e^{-t} \end{bmatrix}.$$

Since these two matrices of functions are the same, $\mathbf{\Phi}(t)$ is a solution matrix.

To check that it is a fundamental matrix, pick $t = 0$ for example. Then

$$\mathbf{\Phi}(0) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

and this matrix has determinant -3, so $\mathbf{\Phi}(t)$ is a fundamental matrix. ◀

Example 6.3.6. Show that

$$\mathbf{\Phi}(t) = \begin{bmatrix} te^{2t} & (t+1)e^{2t} \\ e^{2t} & e^{2t} \end{bmatrix}$$

is a fundamental matrix for the system $\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{y}$.

▶ **Solution.** Again we check that the two conditions of Equation (4) are satisfied. First we calculate $\mathbf{\Phi}'(t)$:

$$\mathbf{\Phi}'(t) = \begin{bmatrix} te^{2t} & (t+1)e^{2t} \\ e^{2t} & e^{2t} \end{bmatrix}' = \begin{bmatrix} (2t+1)e^{2t} & (2t+3)e^{2t} \\ 2e^{2t} & 2e^{2t} \end{bmatrix}.$$

Next we calculate $A(t)\mathbf{\Phi}(t)$ where $A(t) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$:

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} te^{2t} & (t+1)e^{2t} \\ e^{2t} & e^{2t} \end{bmatrix} = \begin{bmatrix} (2t+1)e^{2t} & (2t+3)e^{2t} \\ 2e^{2t} & 2e^{2t} \end{bmatrix}.$$

Since these two matrices of functions are the same, $\mathbf{\Phi}(t)$ is a solution matrix.

Next check the second condition of (4) at $t = 0$:

$$\det \mathbf{\Phi}(0) = \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = -1 \neq 0.$$

Hence $\mathbf{\Phi}(t)$ is a fundamental matrix for $\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{y}$. ◀

Example 6.3.7. Show that

$$\mathbf{\Phi}(t) = \begin{bmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{bmatrix}$$

is a fundamental matrix for the system $\boldsymbol{y}' = A(t)\boldsymbol{y}$ where

$$A(t) = \begin{bmatrix} 0 & 1 \\ -\frac{6}{t^2} & \frac{4}{t} \end{bmatrix}.$$

▶ **Solution.** Note that

$$\mathbf{\Phi}'(t) = \begin{bmatrix} 2t & 3t^2 \\ 2 & 6t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{6}{t^2} & \frac{4}{t} \end{bmatrix} \begin{bmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{bmatrix},$$

while

$$\det \mathbf{\Phi}(1) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = 1 \neq 0.$$

Hence $\mathbf{\Phi}(t)$ is a fundamental matrix. Note that $\mathbf{\Phi}(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which has determinant 0. Why does this not prevent $\mathbf{\Phi}(t)$ from being a fundamental matrix? ◀

In Exercise 29, Page 333 you were asked to verify the product rule for differentiating a product of matrix functions. The rule is

$$(B(t)C(t))' = B'(t)C(t) + B(t)C'(t).$$

Since matrix multiplication is not commutative, it is necessary to be careful of the order. If one of the matrices is *constant*, then the product rule is simpler:

$$(B(t)C)' = B'(t)C$$

since $C' = 0$ for a constant matrix. We apply this observation in the following way. Suppose that $\mathbf{\Phi}_1(t)$ is a fundamental matrix of a homogeneous system $\boldsymbol{y}' = A(t)\boldsymbol{y}$ where $A(t)$ is an $n \times n$ matrix of continuous functions on an interval I. According to Theorem 6.3.4 this means that

$$\mathbf{\Phi}_1'(t) = A(t)\mathbf{\Phi}_1(t) \quad \text{and} \quad \det \mathbf{\Phi}_1(t) \neq 0.$$

Now define a new $n \times n$ matrix of functions $\mathbf{\Phi}_2(t) := \mathbf{\Phi}_1(t)C$ where C is an $n \times n$ constant matrix. Then

$$\mathbf{\Phi}_2'(t) = (\mathbf{\Phi}_1(t)C)' = \mathbf{\Phi}_1'(t)C = A(t)\mathbf{\Phi}_1(t)C = A(t)\mathbf{\Phi}_2(t),$$

so that $\mathbf{\Phi}_2(t)$ is a solution matrix for the homogeneous system $\boldsymbol{y}' = A(t)\boldsymbol{y}$. To determine if $\mathbf{\Phi}_2(t)$ is also a fundamental matrix, it is only necessary to compute the determinant:

$$\det \mathbf{\Phi}_2(t) = \det(\mathbf{\Phi}_1(t)C) = \det \mathbf{\Phi}_1(t) \det C.$$

Since $\det \mathbf{\Phi}_1(t) \neq 0$, it follows that $\det \mathbf{\Phi}_2(t) \neq 0$ if and only if $\det C \neq 0$, i.e., if and only if C is a nonsingular $n \times n$ matrix.

Example 6.3.8. In Example 6.3.5 it was shown that

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix}$$

is a fundamental matrix for the system $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{y}$. Let $C = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Then $\det C = -1/3 \neq 0$ so C is invertible, and hence

$$\mathbf{\Psi}(t) = \mathbf{\Phi}(t)C = \frac{1}{3}\begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} e^{2t} + 2e^{-t} & e^{2t} - e^{-t} \\ 2e^{2t} - 2e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix}$$

is also a fundamental matrix for $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{y}$. Note that $\mathbf{\Psi}(t)$ has the particularly nice feature that its value at $t = 0$ is

$$\mathbf{\Psi}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

the 2×2 identity matrix. □

Example 6.3.9. In Example 6.3.6 it was shown that

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} te^{2t} & (t+1)e^{2t} \\ e^{2t} & e^{2t} \end{bmatrix}$$

is a fundamental matrix for the system $\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{y}$. Let $C = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\det C = -1 \neq 0$ so C is invertible, and hence

$$\boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)C = \begin{bmatrix} te^{2t} & (t+1)e^{2t} \\ e^{2t} & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

is also a fundamental matrix for $\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{y}$. As in the previous example $\boldsymbol{\Psi}(0) = I_2$ is the identity matrix. □

If $\boldsymbol{\Phi}(t)$ is a fundamental matrix for the linear system $\boldsymbol{y}' = A(t)\boldsymbol{y}$ on the interval I and $t_0 \in I$, then $\boldsymbol{\Phi}(t_0)$ is an invertible matrix by Theorem 6.3.4 so if we take $C = (\boldsymbol{\Phi}(t_0))^{-1}$, then

$$\boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)C = \boldsymbol{\Phi}(t)(\boldsymbol{\Phi}(t_0))^{-1}$$

is a fundamental matrix which satisfies the extra condition

$$\boldsymbol{\Psi}(t_0) = \boldsymbol{\Phi}(t_0)(\boldsymbol{\Phi}(t_0))^{-1} = I_n.$$

Hence, we can always arrange for our fundamental matrices to be the identity at the initial point t_0. Moreover, the uniqueness part of the existence and uniqueness theorem insures that there is only one solution matrix satisfying this extra condition. We record this observation in the following result.

Theorem 6.3.10. *If $A(t)$ is a continuous $n \times n$ matrix of functions on an interval I and $t_0 \in I$, then there is an $n \times n$ matrix of functions $\boldsymbol{\Psi}(t)$ such that*

1. *$\boldsymbol{\Psi}(t)$ is a fundamental matrix for the homogeneous linear equation $\boldsymbol{y}' = A(t)\boldsymbol{y}$ and*
2. *$\boldsymbol{\Psi}(t_0) = I_n$,*
3. *Moreover, $\boldsymbol{\Psi}(t)$ is uniquely determined by these two properties.*
4. *If $\boldsymbol{y}_0$ is a constant vector, then $\boldsymbol{y}(t) = \boldsymbol{\Psi}(t)\boldsymbol{y}_0$ is the unique solution of the homogeneous initial value problem $\boldsymbol{y}' = A(t)\boldsymbol{y}$, $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$.*

Proof. Only the last statement was not discussed in the preceding paragraphs. Suppose that $\boldsymbol{\Psi}(t)$ satisfies conditions (1) and (2) and let $\boldsymbol{y}(t) = \boldsymbol{\Psi}(t)\boldsymbol{y}_0$. Then $\boldsymbol{y}(t_0) = \boldsymbol{\Psi}(t_0)\boldsymbol{y}_0 = I_n\boldsymbol{y}_0 = \boldsymbol{y}_0$. Moreover,

$$\boldsymbol{y}'(t) = \boldsymbol{\Psi}'(t)\boldsymbol{y}_0 = A(t)\boldsymbol{\Psi}(t)\boldsymbol{y}_0 = A(t)\boldsymbol{y}(t),$$

so $\boldsymbol{y}(t)$ is a solution of the initial value problem, as required. □

Example 6.3.11. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$.

▶ **Solution.** In Example 6.3.8 we found a fundamental matrix for $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \boldsymbol{y}$ satisfying (1) and (2) of the above theorem, namely

$$\boldsymbol{\Psi}(t) = \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{-t} & e^{2t} - e^{-t} \\ 2e^{2t} - 2e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix}.$$

Hence the unique solution of the initial value problem is

$$\boldsymbol{y}(t) = \boldsymbol{\Psi}(t) \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{-t} & e^{2t} - e^{-t} \\ 2e^{2t} - 2e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} -e^{2t} + 4e^{-t} \\ -2e^{2t} - 4e^{-t} \end{bmatrix}.$$

◀

Example 6.3.12. Solve the initial value problem $\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

▶ **Solution.** In Example 6.3.9 we found a fundamental matrix for $\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{y}$ satisfying (1) and (2) of the above theorem, namely

$$\boldsymbol{\Psi}(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}.$$

Hence the solution of the initial value problem is

$$\boldsymbol{y}(t) = \boldsymbol{\Psi}(t) \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} (3t-2)e^{2t} \\ 3e^{2t} \end{bmatrix}.$$

◀

We conclude this section by observing that for a constant matrix function $A(t) = A$, at least in principle, it is easy to describe the fundamental matrix $\mathbf{\Psi}(t)$ from Theorem 6.3.10. It is in fact the matrix function we have already encountered in the last section, i.e., the matrix exponential e^{At}. Recall that e^{At} (for a constant matrix A) is defined by substituting A for a in the Taylor series expansion of e^{at}:

$$(*) \qquad e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n + \cdots$$

We have already observed (but not proved) that the series on the right hand side of $(*)$ converges to a well defined matrix function for all matrices A. Let $\mathbf{\Psi}(t) = e^{At}$. If we set $t = 0$ in the series we obtain $\mathbf{\Psi}(0) = e^{A0} = I_n$ and if we differentiate the series terms by term (which can be shown to be a valid operation), we get

$$\begin{aligned}
\mathbf{\Psi}'(t) &= \frac{d}{dt}e^{At} \\
&= 0 + A + A^2t + \frac{1}{2}A^3t^2 + \cdots + \frac{1}{(n-1)!}A^nt^{n-1} + \cdots \\
&= A\left(I_n + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}t^{n-1} + \cdots\right) \\
&= Ae^{At} \\
&= A\mathbf{\Psi}(t).
\end{aligned}$$

Thus we have shown that $\mathbf{\Psi}(t) = e^{At}$ satisfies the first two properties of Theorem 6.3.10, and hence we have arrived at the important result:

Theorem 6.3.13. *Suppose A is an $n \times n$ constant matrix.*

1. *A fundamental matrix for the linear homogeneous problem $\boldsymbol{y}' = A\boldsymbol{y}$ is $\mathbf{\Psi}(t) = e^{At}$.*
2. *If $\boldsymbol{y}_0$ is a constant vector, then the unique solution of the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is*

$$\boxed{\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0.} \qquad (5)$$

3. *If $\mathbf{\Phi}(t)$ is any fundamental matrix for the problem $\boldsymbol{y}' = A\boldsymbol{y}$, then*

$$\boxed{e^{At} = \mathbf{\Phi}(t)\left(\mathbf{\Phi}(0)\right)^{-1}.} \qquad (6)$$

Example 6.3.14. 1. From the calculations in Example 6.3.8 we conclude that if $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, then

$$e^{At} = \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{-t} & e^{2t} - e^{-t} \\ 2e^{2t} - 2e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix}.$$

2. From the calculations in Example 6.3.9 we conclude that if $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ then

$$e^{At} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}.$$

What we have seen in this section is that if we can solve $\boldsymbol{y}' = A\boldsymbol{y}$ (where A is a constant matrix), then we can find e^{At} by Equation (6), and conversely, if we can find e^{At} by some method, then we can find all solutions of $\boldsymbol{y}' = A\boldsymbol{y}$ by means of Equation (5). Over the next few sections we will learn a couple of different methods for calculating e^{At}.

Exercises

1. For each of the following pairs of matrix functions $\boldsymbol{\Phi}(t)$ and $A(t)$, determine if $\boldsymbol{\Phi}(t)$ is a fundamental matrix for the system $\boldsymbol{y}' = A(t)\boldsymbol{y}$. It may be useful to review Examples 4.3.5 – 4.3.7.

	$\mathbf{\Phi}(t)$	$A(t)$
(a)	$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
(b)	$\begin{bmatrix} \cos t & \sin t \\ -\sin(t+\pi/2) & \cos(t+\pi/2) \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
(c)	$\begin{bmatrix} e^{-t} & e^{2t} \\ e^{-t} & 4e^{2t} \end{bmatrix}$	$\begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$
(d)	$\begin{bmatrix} e^{-t}-e^{2t} & e^{2t} \\ e^{-t}-4e^{2t} & 4e^{2t} \end{bmatrix}$	$\begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$
(e)	$\begin{bmatrix} e^{t} & e^{2t} \\ e^{3t} & e^{4t} \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
(f)	$\begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}$	$\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix}$
(g)	$\begin{bmatrix} 3e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix}$	$\begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$
(h)	$\begin{bmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & te^{3t} \end{bmatrix}$	$\begin{bmatrix} 1 & -4 \\ 1 & 5 \end{bmatrix}$
(i)	$\begin{bmatrix} \sin(t^2/2) & \cos(t^2/2) \\ \cos(t^2/2) & -\sin(t^2/2) \end{bmatrix}$	$\begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}$
(j)	$\begin{bmatrix} 1+t^2 & 3+t^2 \\ 1-t^2 & -1-t^2 \end{bmatrix}$	$\begin{bmatrix} t & t \\ -t & -t \end{bmatrix}$
(k)	$\begin{bmatrix} e^{t^2/2} & e^{-t^2/2} \\ e^{t^2/2} & -e^{-t^2/2} \end{bmatrix}$	$\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}$

2. For each of the matrices A in parts (a), (c), (f), (g), (h) of Exercise 1:

   (a) Find a fundamental matrix $\mathbf{\Psi}(t)$ for the system $\boldsymbol{y}' = A\boldsymbol{y}$ satisfying the condition $\mathbf{\Psi}(0) = I_2$. (See Examples 4.3.8 and 4.3.9.)

   (b) Solve the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

   (c) Find e^{At}.

3. For each of the matrices $A(t)$ in parts (i), (j) and (k) of Exercise 1:

   (a) Find a fundamental matrix $\mathbf{\Psi}(t)$ for the system $\boldsymbol{y}' = A(t)\boldsymbol{y}$ satisfying the condition $\mathbf{\Psi}(0) = I_2$.

   (b) Solve the initial value problem $\boldsymbol{y}' = A(t)\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

   (c) Is $e^{A(t)t} = \mathbf{\Psi}(t)$? Explain.

4. In each problem below determine whether the given functions are linearly independent.

   (a) $\mathbf{y_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{y_2} = \begin{bmatrix} t \\ -t \end{bmatrix}$.

   (b) $\mathbf{y_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{y_2} = \begin{bmatrix} t \\ t \end{bmatrix}$.

   (c) $\mathbf{y_1} = \begin{bmatrix} te^t \\ e^t \end{bmatrix} \quad \mathbf{y_2} = \begin{bmatrix} e^{-t} \\ te^{-t} \end{bmatrix}$.

   (d) $\mathbf{y_1} = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} \quad \mathbf{y_2} = \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} \quad \mathbf{y_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

6.4 Constant Coefficient Homogeneous Systems

In previous sections we studied some of the basic properties of the homogeneous linear system of differential equations

$$\boldsymbol{y}' = A(t)\boldsymbol{y}. \tag{1}$$

In the case of a constant coefficient system, i.e., $A(t) = A =$ a constant matrix, this analysis culminated in Theorem 6.3.13 which states that a fundamental matrix for $\boldsymbol{y}' = A\boldsymbol{y}$ is the matrix exponential function e^{At} and the unique solution of the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0.$$

That is, the solution of the initial value problem is obtained by multiplying the fundamental matrix e^{At} by the initial value vector $\boldsymbol{y}_0$. The problem of how to compute e^{At} for a particular constant matrix A was not addressed, except for a few special cases where e^{At} could be computed directly from the series definition of e^{At}. In this section we will show how to use the Laplace transform to solve the constant coefficient homogeneous system $\boldsymbol{y}' = A\boldsymbol{y}$ and in the process we will arrive at a Laplace transform formula for e^{At}.

As we have done previously, we will do our calculations in detail for the case of a constant coefficient linear system where the coefficient matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 constant matrix so that Equation (1) becomes

$$\begin{aligned} y_1' &= ay_1 + by_2 \\ y_2' &= cy_1 + dy_2. \end{aligned} \tag{2}$$

The calculations are easily extended to systems with more than 2 unknown functions. According to the existence and uniqueness theorem (Theorem 6.2.2) there is a solution $\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ for system (2), and we assume that the functions $y_1(t)$ and $y_2(t)$ have Laplace transforms. From Chapter 2, we know that this is a relatively mild restriction on these functions, since, in particular, all functions of exponential growth have Laplace transforms. Our strategy will be to use the Laplace transform of the system (2) to determine what the solution must be.

Let $Y_1(s) = \mathcal{L}(y_1)$ and $Y_2(s) = \mathcal{L}(y_2)$. Applying the Laplace transform to each equation in system (2) and using the formulas from Table C.2 gives a system of algebraic equations

$$\begin{aligned} sY_1(s) - y_1(0) &= aY_1(s) + bY_2(s) \\ sY_2(s) - y_2(0) &= cY_1(s) + dY_2(s). \end{aligned} \tag{3}$$

Letting $\boldsymbol{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$, the system (3) can be written compactly in matrix form as

$$s\boldsymbol{Y}(s) - \boldsymbol{y}(0) = A\boldsymbol{Y}(s)$$

which is then easily rewritten as the matrix equation

$$(sI - A)\,\boldsymbol{Y}(s) = \boldsymbol{y}(0). \tag{4}$$

If the matrix $sI - A$ is invertible, then we may solve Equation (4) for $\boldsymbol{Y}(s)$, and then apply the inverse Laplace transform to the entries of $\boldsymbol{Y}(s)$ to find the unknown functions $\boldsymbol{y}(t)$. But

$$sI - A = \begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix} \tag{5}$$

so $p(s) = \det(sI - A) = (s-a)(s-b) - bc = s^2 - (a+d)s + (ad-bc) = s^2 - \mathrm{Tr}(A)s + \det(A)$. Hence $p(s)$ is a nonzero polynomial function of degree 2, so that the matrix $sI - A$ is invertible as a matrix of rational functions, although one should note that for certain (the ≤ 2 roots of $p(s)$) values of s the numerical matrix will not be invertible. For the purposes of Laplace transforms, we are only interested in the inverse of $sI - A$ as a matrix of rational functions. Hence we may solve Equation (4) for $\boldsymbol{Y}(s)$ to get

$$\boldsymbol{Y}(s) = (sI - A)^{-1}\,\boldsymbol{y}(0). \tag{6}$$

Now

$$(sI - A)^{-1} = \frac{1}{p(s)} \begin{bmatrix} s-d & b \\ c & s-a \end{bmatrix} \tag{7}$$

so let $\boldsymbol{Z}_1(s) = \dfrac{1}{p(s)}\begin{bmatrix} s-d \\ c \end{bmatrix}$ and $\boldsymbol{Z}_2(s) = \dfrac{1}{p(s)}\begin{bmatrix} b \\ s-a \end{bmatrix}$ be the first and second columns of $(sI - A)^{-1} := \boldsymbol{Z}(s)$, respectively. Since each entry of $\boldsymbol{Z}_1(s)$ and $\boldsymbol{Z}_2(s)$ is a rational function of s with denominator the quadratic polynomial $p(s)$, the analysis of inverse Laplace transforms of rational functions of s with quadratic denominator from Section 3.3 applies to show that each entry of

$$\boldsymbol{z}_1(t) = \mathcal{L}^{-1}\boldsymbol{Z}_1(s) = \begin{bmatrix} \mathcal{L}^{-1}\left(\dfrac{s-d}{p(s)}\right) \\ \\ \mathcal{L}^{-1}\left(\dfrac{c}{p(s)}\right) \end{bmatrix} \quad \text{and} \quad \boldsymbol{z}_2(t) = \mathcal{L}^{-1}\boldsymbol{Z}_2(s) = \begin{bmatrix} \mathcal{L}^{-1}\left(\dfrac{b}{p(s)}\right) \\ \\ \mathcal{L}^{-1}\left(\dfrac{s-a}{p(s)}\right) \end{bmatrix}$$

will be of the form

1. $c_1e^{r_1t} + c_2e^{r_2t}$ if $p(s)$ has distinct real roots $r_1 \neq r_2$;
2. $c_1e^{rt} + c_2te^{rt}$ if $p(s)$ has a double root r; or
3. $c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta t$ if $p(s)$ has complex roots $\alpha \pm i\beta$,

where c_1 and c_2 are appropriate constants. Equation (6) shows that

$$\begin{aligned} \boldsymbol{Y}(s) &= \boldsymbol{Z}(s)\boldsymbol{y}(0) \\ &= y_1(0)\boldsymbol{Z}_1(s) + y_2(0)\boldsymbol{Z}_2(s), \end{aligned} \tag{8}$$

and by applying the inverse Laplace transform we conclude that the solution to Equation (2) is

$$\boldsymbol{y}(t) = y_1(0)\boldsymbol{z}_1(t) + y_2(0)\boldsymbol{z}_2(t). \tag{9}$$

If we let

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{z}_1(t) & \boldsymbol{z}_2(t) \end{bmatrix} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right), \tag{10}$$

then Equation (9) for the solution $\boldsymbol{y}(t)$ of system (2) has a particularly nice and useful matrix formulation:

$$\boxed{\boldsymbol{y}(t) = \boldsymbol{z}(t)\boldsymbol{y}(0).} \tag{11}$$

Before analyzing Equation (11) further to extract theoretical conclusions, we will first see what the solutions look like in a few numerical examples.

Example 6.4.1. Find all solutions of the constant coefficient homogeneous linear system:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 4y_1. \end{aligned} \tag{12}$$

▶ **Solution.** In this system the coefficient matrix is $A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$. Thus $sI - A = \begin{bmatrix} s & -1 \\ -4 & s \end{bmatrix}$ so that $p(s) = \det(sI - A) = s^2 - 4$ and

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s}{s^2-4} & \dfrac{1}{s^2-4} \\[2ex] \dfrac{4}{s^2-4} & \dfrac{s}{s^2-4} \end{bmatrix}. \tag{13}$$

Since

$$\frac{1}{s^2-4} = \frac{1}{4}\left(\frac{1}{s-2} - \frac{1}{s+2}\right) \quad \text{and} \quad \frac{s}{s^2-4} = \frac{1}{2}\left(\frac{1}{s-2} + \frac{1}{s+2}\right),$$

we conclude from our Laplace transform formulas (Table C.2) that the matrix $\boldsymbol{z}(t)$ of Equation (10) is

$$\boldsymbol{z}(t) = \begin{bmatrix} \dfrac{1}{2}\left(e^{2t} + e^{-2t}\right) & \dfrac{1}{4}\left(e^{2t} - e^{-2t}\right) \\[2ex] e^{2t} - e^{-2t} & \dfrac{1}{2}\left(e^{2t} + e^{-2t}\right) \end{bmatrix}. \tag{14}$$

Hence, the solution of the system (12) is

$$\begin{aligned} \boldsymbol{y}(t) &= \begin{bmatrix} \dfrac{1}{2}\left(e^{2t} + e^{-2t}\right) & \dfrac{1}{4}\left(e^{2t} - e^{-2t}\right) \\[2ex] e^{2t} - e^{-2t} & \dfrac{1}{2}\left(e^{2t} + e^{-2t}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1}{2}c_1 + \dfrac{1}{4}c_2 \\[2ex] c_1 + \dfrac{1}{2}c_2 \end{bmatrix} e^{2t} + \begin{bmatrix} \dfrac{1}{2}c_1 - \dfrac{1}{4}c_2 \\[2ex] -c_1 + \dfrac{1}{2}c_2 \end{bmatrix} e^{-2t}, \end{aligned} \tag{15}$$

where $\boldsymbol{y}(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Let's check that we have, indeed, found a solution to the system of differential equations (2). From Equation (15) we see that

$$y_1(t) = \left(\frac{1}{2}c_1 + \frac{1}{4}c_2\right)e^{2t} + \left(\frac{1}{2}c_1 - \frac{1}{4}c_2\right)e^{-2t},$$

and

$$y_2(t) = \left(c_1 + \frac{1}{2}c_2\right)e^{2t} + \left(-c_1 - \frac{1}{2}c_2\right)e^{-2t}.$$

Thus $y_1'(t) = y_2(t)$ and $y_2'(t) = 4y_1(t)$, which is what it means to be a solution of system (12).

The solution to system (12) with initial conditions $y_1(0) = 1$, $y_2(0) = 0$ is

$$\boldsymbol{y}_1(t) = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix} e^{-2t}$$

while the solution with initial conditions $y_1(0) = 0$, $y_2(0) = 1$ is

$$\boldsymbol{y}_2(t) = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} e^{2t} + \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \end{bmatrix} e^{-2t}.$$

The solution with initial conditions $y_1(0) = c_1$, $y_2(0) = c_2$ can then be written

$$\boldsymbol{y}(t) = c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t),$$

that is, every solution $\boldsymbol{y}$ of system (12) is a linear combination of the two particular solution $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$. Note, in particular, that $\boldsymbol{y}_3(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$ is a solution (with $c_1 = 1$, $c_2 = 2$), while $\boldsymbol{y}_4 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$ is also a solution (with $c_1 = 1$, $c_2 = -2$). The solutions $\boldsymbol{y}_3(t)$ and $\boldsymbol{y}_4(t)$ are notably simple solutions in that each of these solutions is of the form

$$\boldsymbol{y}(t) = \boldsymbol{v}e^{at} \tag{16}$$

where $\boldsymbol{v} \in \mathbb{R}^2$ is a constant vector and a is a scalar. Note that

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

That is, the vectors $\boldsymbol{v}$ and scalars a such that $\boldsymbol{y}(t) = \boldsymbol{v}e^{at}$ is a solution to $\boldsymbol{y}' = A\boldsymbol{y}$ are related by the algebraic equation

$$A\boldsymbol{v} = a\boldsymbol{v}. \tag{17}$$

A vector-scalar pair $(\boldsymbol{v}, a)$ which satisfies Equation 17 is known as a **eigenvector-eigenvalue** pair for the matrix A. Finally, compare these two solutions $\boldsymbol{y}_3(t)$ and $\boldsymbol{y}_4(t)$ of the matrix differential equation $\boldsymbol{y}' = A\boldsymbol{y}$ with the solution of the scalar differential equation $y' = ay$, which we recall (see Section 1.3) is $y(t) = ve^{at}$ where $v = y(0) \in \mathbb{R}$ is a scalar. In both cases one gets either a scalar or a vector multiplied by a pure exponential function e^{at}. ◀

Example 6.4.2. Find all solutions of the linear homogeneous system

$$\begin{array}{rcrcr} y_1' & = & y_1 & + & y_2 \\ y_2' & = & -4y_1 & - & 3y_2. \end{array} \tag{18}$$

► **Solution.** For this system, the coefficient matrix is $A = \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}$. We will solve this equation by using Equation (11). Form the matrix

$$sI - A = \begin{bmatrix} s-1 & -1 \\ 4 & s+3 \end{bmatrix}.$$

Then $p(s) = \det(sI - A) = (s-1)(s+3) + 4 = (s+1)^2$, so that

$$\begin{aligned} (sI - A)^{-1} &= \frac{1}{(s+1)^2}\begin{bmatrix} s+3 & 1 \\ -4 & s-1 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1}{s+1} + \dfrac{2}{(s+1)^2} & \dfrac{1}{(s+1)^2} \\ \\ \dfrac{-4}{(s+1)^2} & \dfrac{1}{s+1} - \dfrac{2}{(s+1)^2} \end{bmatrix}. \end{aligned} \tag{19}$$

Thus the matrix $\boldsymbol{z}(t)$ from Equation (10) is, using the inverse Laplace formulas from Table C.2

$$\boldsymbol{z}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = \begin{bmatrix} e^{-t} + 2te^{-t} & te^{-t} \\ -4te^{-t} & e^{-t} - 2te^{-t} \end{bmatrix}.$$

The general solution to system (18) is therefore

$$\boldsymbol{y}(t) = \boldsymbol{z}(t)\boldsymbol{y}(0) = c_1\begin{bmatrix} e^{-t} + 2te^{-t} \\ -4te^{-t} \end{bmatrix} + c_2\begin{bmatrix} te^{-1} \\ e^{-t} - 2te^{-t} \end{bmatrix}. \tag{20}$$

Taking $c_1 = 1$ and $c_2 = -2$ in this equation gives a solution

$$\boldsymbol{y}_1(t) = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t},$$

which is a solution of the form $\boldsymbol{y}(t) = \boldsymbol{v}e^{at}$ where $\boldsymbol{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a constant vector and $a = -1$ is a scalar. Note that $(\boldsymbol{v}, -1)$ is an eigenvector-eigenvalue pair for the matrix A (see Example 6.4.1). That is,

$$A\boldsymbol{v} = A \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1) \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

◀

Example 6.4.3. Find the solution of the linear homogeneous initial value problem:

$$\begin{array}{rcrcr} y_1' & = & y_1 & + & 2y_2 \\ y_2' & = & -2y_1 & + & y_2 \end{array}, \quad y_1(0) = c_1,\ y_2(0) = c_2. \tag{21}$$

▶ **Solution.** For this system, the coefficient matrix is $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. We will solve this equation by using Equation (11), as was done for the previous examples. Form the matrix

$$sI - A = \begin{bmatrix} s-1 & -2 \\ 2 & s-1 \end{bmatrix}.$$

Then $p(s) = \det(sI - A) = (s-1)^2) + 4$, so that

$$\begin{aligned} (sI - A)^{-1} &= \frac{1}{(s-1)^2 + 4} \begin{bmatrix} s-1 & 2 \\ -2 & s-1 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{s-1}{(s-1)^2+4} & \dfrac{2}{(s-1)^2+4} \\ \dfrac{-2}{(s-1)^2+4} & \dfrac{s-1}{(s-1)^2+4} \end{bmatrix}. \end{aligned}$$

Hence, using the inverse Laplace transform formulas from Table C.2, the matrix $\boldsymbol{z}(t)$ of Equation (10) is

$$\boldsymbol{z}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = \begin{bmatrix} e^t \cos 2t & e^t \sin 2t \\ -e^t \sin 2t & e^t \cos 2t \end{bmatrix},$$

and the solution of system (21) is

$$\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t \cos 2t + c_2 e^t \sin 2t \\ -c_1 e^t \sin 2t + c_2 e^t \cos 2t \end{bmatrix}.$$

◀

Now we return briefly to the theoretical significance of Equation (11). According to the analysis leading to (11), the unique solution of the initial value problem

$$(*) \qquad \boldsymbol{y}' = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0,$$

where A is a 2×2 constant matrix, is

$$\boldsymbol{y}(t) = \boldsymbol{z}(t)\boldsymbol{y}(0) = \boldsymbol{z}(t)\boldsymbol{y}_0,$$

where

$$\boldsymbol{z}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right).$$

But according to Theorem 6.3.13, the unique solution of the initial value problem (*) is

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0.$$

These two descriptions of $\boldsymbol{y}(t)$ give an equality of matrix functions

$$(**) \qquad \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\boldsymbol{y}_0 = \boldsymbol{z}(t)\boldsymbol{y}_0 = e^{At}\boldsymbol{y}_0$$

which holds for *all* choices of the constant vector $\boldsymbol{y}_0$. But if C is a 2×2 matrix then $C\boldsymbol{e}_1 = C\begin{bmatrix}1\\0\end{bmatrix}$ is the first column of C and $C\boldsymbol{e}_2 = C\begin{bmatrix}0\\1\end{bmatrix}$ is the second column of C (check this!). Thus, if B and C are two 2×2 matrices such that $B\boldsymbol{e}_i = C\boldsymbol{e}_i$ for $i = 1$, 2, then $B = C$ (since column i of B = column i of C for $i = 1$, 2). Taking $\boldsymbol{y}_0 = \boldsymbol{e}_i$ for $i = 1$, 2, and applying this observation to the matrices of $(**)$, we arrive at the following result:

Theorem 6.4.4. *If A is a 2×2 constant matrix, then*

$$e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right). \tag{22}$$

Example 6.4.5. From the calculations of $\mathcal{L}^{-1}\left((sI - A)^{-1}\right)$ done in Examples 6.4.1, 6.4.2 and 6.4.3 this theorem gives the following values of e^{At}:

A	e^{At}
$\begin{bmatrix}0 & 1\\4 & 0\end{bmatrix}$	$\begin{bmatrix}\frac{1}{2}\left(e^{2t} + e^{-2t}\right) & \frac{1}{4}\left(e^{2t} - e^{-2t}\right)\\ e^{2t} - e^{-2t} & \frac{1}{2}\left(e^{2t} + e^{-2t}\right)\end{bmatrix}$
$\begin{bmatrix}1 & 1\\-4 & -3\end{bmatrix}$	$\begin{bmatrix}e^{-t} + 2te^{-t} & te^{-t}\\ -4te^{-t} & e^{-t} - 2te^{-t}\end{bmatrix}$
$\begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}$	$\begin{bmatrix}e^{t}\cos 2t & e^{t}\sin 2t\\ -e^{t}\sin 2t & e^{t}\cos 2t\end{bmatrix}$

While our derivation of the formula for e^{At} in Theorem 6.4.4 was done for 2×2 matrices, the formula remains valid for arbitrary *constant* $n \times n$ matrices A, and moreover, once one can guess that there is a relationship between e^{At} and $\mathcal{L}^{-1}\left((sI - A)^{-1}\right)$, it is a simple matter to verify it by computing the Laplace transform of the matrix function e^{At}. This computation is, in fact, almost the same as the computation of $\mathcal{L}(e^{at})$ in Example 2.1.4.

Theorem 6.4.6. *If A is an $n \times n$ constant matrix (whose entries can be either real numbers or complex numbers), then*

$$\boxed{e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right).} \tag{23}$$

Proof. Note that if B is an $n \times n$ invertible matrix (of constants), then

$$\frac{d}{dt}\left(B^{-1}e^{Bt}\right) = B^{-1}\frac{d}{dt}e^{Bt} = B^{-1}Be^{Bt} = e^{Bt},$$

so that

$$(\dagger) \qquad \int_1^t e^{B\tau}\,d\tau = B^{-1}(e^{Bt} - I).$$

Note that this is just the matrix analog of the integration formula

$$\int_0^t e^{b\tau}\,d\tau = b^{-1}(e^{bt} - 1).$$

Now just mimic the scalar calculation from Example 2.1.4, and note that formula (†) will be applied with $B = A - sI$, where, as usual, I will denote the $n \times n$ identity matrix.

$$\begin{aligned}
\mathcal{L}\left(e^{At}\right)(s) &= \int_0^\infty e^{At}e^{-st}\,dt \\
&= \int_0^\infty e^{At}e^{-stI}\,dt \\
&= \int_0^\infty e^{(A-sI)t}\,dt \\
&= \lim_{N\to\infty}\int_0^N e^{(A-sI)t}\,dt \\
&= \lim_{N\to\infty}(A - sI)^{-1}\left(e^{(A-sI)N} - I\right) \\
&= (sI - A)^{-1}.
\end{aligned}$$

The last equality is justified since $\lim_{N\to\infty} e^{(A-sI)N} = 0$ if s is large enough. This fact, analogous to the fact that $e^{(a-s)t}$ converges to 0 as $t \to \infty$ provided $s > a$, will not be proved. □

Example 6.4.7. Compute e^{At} for the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ -3 & 1 & 3 \\ 3 & -3 & 1 \end{bmatrix},$$

and using the calculation of e^{At}, solve the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

► **Solution.** According to Theorem 6.4.6, $e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)$, so we need to begin by computing $(sI - A)^{-1}$, which is most conveniently done (by hand) by using the adjoint formula for a matrix inverse (see Corollary 5.4.8). Recall that this formula says that if B is an $n \times n$ matrix with $\det B \neq 0$, then $B^{-1} = (\det B)^{-1}[C_{ij}]$ where the term C_{ij} is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the j^{th} row and i^{th} column from B. We apply this with $B = sI - A$. Start by calculating

$$p(s) = \det(sI - A) = \det \begin{bmatrix} s-1 & 3 & -3 \\ 3 & s-1 & -3 \\ -3 & 3 & s-1 \end{bmatrix} = (s-1)(s+2)(s-4).$$

In particular, $sI - A$ is invertible whenever $p(s) \neq 0$, i.e., whenever $s \neq 1$, -2, or 4. Then a tedious, but straightforward calculation, gives

$$\begin{aligned}
(sI - A)^{-1} &= \frac{1}{p(s)} \begin{bmatrix} (s-1)^2 + 9 & -3(s+2) & 3(s-4) \\ -3(s-4) & (s-1)^2 - 9 & 3(s-4) \\ 3(s+2) & -3(s+2) & (s-1)^2 - 9 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{(s-1)^2+9}{p(s)} & \dfrac{-3}{(s-1)(s+4)} & \dfrac{3}{(s-1)(s+2)} \\ \dfrac{-3}{(s-1)(s+2)} & \dfrac{1}{s-1} & \dfrac{3}{(s-1)(s+2)} \\ \dfrac{3}{(s-1)(s-4)} & \dfrac{-3}{(s-1)(s-4)} & \dfrac{1}{s-1} \end{bmatrix} \\
&= \begin{bmatrix} -\dfrac{1}{s-1} + \dfrac{1}{s+2} + \dfrac{1}{s-4} & \dfrac{1}{s-1} - \dfrac{1}{s-4} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ \dfrac{-1}{s-1} + \dfrac{1}{s+2} & \dfrac{1}{s-1} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ \dfrac{-1}{s-1} + \dfrac{1}{s-4} & \dfrac{1}{s-1} - \dfrac{1}{s-4} & \dfrac{1}{s-1} \end{bmatrix}.
\end{aligned}$$

By applying the inverse Laplace transform to each function in the last matrix gives

$$e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = \begin{bmatrix} -e^t + e^{-2t} + e^{4t} & e^t - e^{4t} & e^t - e^{-2t} \\ -e^t + e^{-2t} & e^t & e^t - e^{-2t} \\ -e^t + e^{4t} & e^t - e^{4t} & e^t \end{bmatrix}.$$

Then the solution of the initial value problem is given by $\boldsymbol{y}(t) = e^{At}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t$. ◀

Remark 6.4.8. Most of the examples of numerical systems which we have discussed in this section are first order constant coefficient linear systems with two unknown functions, i.e. $n = 2$ in Definition 6.1.3. Nevertheless, the same analysis works for first order constant coefficient linear systems in any number of unknown functions, i.e. arbitrary n. Specifically, Equations (6) and (11) apply to give the Laplace transform $\boldsymbol{Y}(s)$ and the solution function $\boldsymbol{y}(t)$ for the constant coefficient homogeneous linear system

$$\boldsymbol{y}' = A\boldsymbol{y}$$

where A is an $n \times n$ constant matrix. The practical difficulty in carrying out this program is in calculating $(sI - A)^{-1}$. This can be done by programs like Mathematica, MatLab, or Maple if n is not too large. But even if the calculations of specific entries in the matrix $(sI - A)^{-1}$ are difficult, one can extract useful theoretical information concerning the nature of the solutions of $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}(t)$ from the formulas like Equation (11) and from theoretical algebraic descriptions of the inverse matrix $(sI - A)^{-1}$.

Exercises

For each of the following matrices A, (a) find the matrix $\boldsymbol{z}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)$ from Equation (4.4.10) and (b) find the general solution of the homogeneous system $\boldsymbol{y}' = A\boldsymbol{y}$. It will be useful to review the calculations in Examples 4.4.1 – 4.4.3.

1. $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$
4. $\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$
5. $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$
6. $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$
7. $\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$
8. $\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$
9. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
10. $\begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$
11. $\begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
12. $\begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{bmatrix}$
13. $\begin{bmatrix} -2 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & -2 & -1 \end{bmatrix}$
14. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$
15. $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

6.5 Computing e^{At}

In this section we will present a variant of a technique due to Fulmer[2] for computing the matrix exponential e^{At}. It is based on the knowledge of what *type* of functions are included in the individual entries of e^{At}. This knowledge is derived from our understanding of the Laplace transform table and the fundamental formula

$$e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)$$

which was proved in Theorem 6.4.6.

To get started, assume that A is an $n \times n$ constant matrix. The matrix $sI - A$ is known as the **characteristic matrix** of A and its determinant

$$p(s) := \det(sI - A)$$

is known as the **characteristic polynomial** of A. The following are some basic properties of $sI - A$ and $p(s)$ which are easily derived from the properties of determinants in Section 5.4.

1. The polynomial $p(s)$ has degree n, when A is an $n \times n$ matrix.
2. The characteristic matrix $sI - A$ is invertible except when $p(s) = 0$.

---

[2]Edward P. Fulmer, Computation of the Matrix Exponential, *American Mathematical Monthly*, **82** (1975) 156–159.

3. Since $p(s)$ is a polynomial of degree n, it has at most n roots (exactly n if multiplicity of roots and complex roots are considered). The roots of $p(s)$ are called the **eigenvalues** of A.

4. The inverse of $sI - A$ is given by the adjoint formula (Corollary 5.4.8)

$$(*) \qquad (sI - A)^{-1} = \frac{1}{p(s)}[C_{ij}(s)] = \left[\frac{C_{ij}(s)}{p(s)}\right]$$

where $C_{ij}(s)$ is $(-1)^{i+j}$ times the determinant of the matrix obtained from $sI - A$ by deleting the i^{th} column and j^{th} row. For example, if $n = 2$ then we get the formula

$$\begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix}^{-1} = \frac{1}{p(s)} \begin{bmatrix} s-d & b \\ c & s-a \end{bmatrix}$$

which we used in Section 6.4.

5. The functions $C_{ij}(s)$ appearing in $(*)$ are polynomials of degree at most $n - 1$. Therefore, the entries

$$p_{ij}(s) = \frac{C_{ij}(s)}{p(s)}$$

of $(sI - A)^{-1}$ are *proper rational functions* with denominator of degree n.

6. Since

$$e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = \left[\mathcal{L}^{-1}\left(\frac{C_{ij}(s)}{p(s)}\right)\right],$$

the *form* of the functions

$$h_{ij}(t) = \mathcal{L}^{-1}\left(\frac{C_{ij}(s)}{p(s)}\right),$$

which are the individual entries of the matrix exponential e^{At}, are completely determined by the roots of $p(s)$ and their multiplicities via the analysis of inverse Laplace transforms of rational functions as described in Section 2.3.

7. Suppose that r is an eigenvalue of A of multiplicity k. That is, r is a root of the characteristic polynomial $p(s)$ and $(s - r)^k$ divides $p(s)$, but no higher power of $s - r$ divides $p(s)$. We distinguish two cases:

Case 1: The eigenvalue r is real.

In this case r will contribute a linear combination of the functions

$$(*_{\text{real}}) \qquad e^{rt}, \quad te^{rt}, \quad \cdots, \quad t^{k-1}e^{rt}$$

to each h_{ij}.

Case 2: The eigenvalue $r = \alpha + i\beta$ has nonzero imaginary part $\beta \neq 0$.
In this case $r = \alpha + i\beta$ and its complex conjugate $\overline{r} = \alpha - i\beta$ will contribute a linear combination of the functions

$$(*_{\text{Imag.}}) \qquad \begin{array}{llll} e^{\alpha t}\cos\beta t, & te^{\alpha t}\cos\beta t, & \cdots, & t^{k-1}e^{\alpha t}\cos\beta t \\ e^{\alpha t}\sin\beta t, & te^{\alpha t}\sin\beta t, & \cdots, & t^{k-1}e^{\alpha t}\sin\beta t \end{array}$$

to each h_{ij}.

8. The total number of functions listed in $(*_{\text{real}})$ and $(*_{\text{Imag.}})$ counting all eigenvalues is $n = \deg p(s)$. If we let $\phi_1, \ldots, \phi_n$ be these n functions, then it follows from our analysis above, that each entry $h_{ij}(t)$ can be written as a linear combination

$$(*) \qquad h_{ij}(t) = m_{ij1}\phi_1(t) + \cdots + m_{ijn}\phi_n(t)$$

of $\phi_1, \ldots, \phi_n$. We will define an $n \times n$ matrix $M_k = [m_{ijk}]$ whose ij^{th} entry is the coefficient of $\phi_k(t)$ in the expansion of $h_{ij}(t)$ in $(*)$. Then we have a matrix equation expressing this linear combination relation:

$$(**) \qquad e^{At} = [h_{ij}(t)] = M_1\phi_1(t) + \cdots + M_n\phi(t).$$

Example 6.5.1. As a specific example of the decomposion given by $(**)$, consider the matrix e^{At} from Example 6.4.7:

$$e^{At} = \begin{bmatrix} -e^t + e^{-2t} + e^{4t} & e^t - e^{4t} & e^t - e^{-2t} \\ -e^t + e^{-2t} & e^t & e^t - e^{-2t} \\ -e^t + e^{4t} & e^t - e^{4t} & e^t \end{bmatrix}.$$

In this case (refer to Example 6.4.7 for details), $p(s) = (s-1)(s+2)(s-4)$ so the eigenvalues are 1, -2 and 4 and the basic functions $\phi_i(t)$ are $\phi_1(t) = e^t$, $\phi_2(t) = e^{-2t}$ and $\phi_3(t) = e^{4t}$. Then $(**)$ is the identity

$$e^{At} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} e^t + \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} e^{4t},$$

where

$$M_1 = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

With the notational preliminaries out of the way, we can give the variation on Fulmer's algorithm for e^{At}.

Algorithm 6.5.2 (Fulmer's method). The following procedure will compute e^{At} where A is a given $n \times n$ constant matrix.

1. Compute $p(s) = \det(sI - A)$.
2. Find all roots and multiplicities of the roots of $p(s)$.
3. From the above observations we have

$$(\ddagger) \qquad e^{At} = M_1\phi_1(t) + \cdots + M_n\phi_n(t),$$

where M_i $i = 1, \ldots, n$ are $n \times n$ matrices. We need to find these matrices.

By taking derivatives we obtain a system of linear equations (with matrix coefficients)

$$\begin{aligned} e^{At} &= M_1\phi_1(t) + \cdots + M_n\phi_n(t) \\ Ae^{At} &= M_1\phi_1'(t) + \cdots + M_n\phi_n'(t) \\ &\vdots \\ A^{n-1}e^{At} &= M_1\phi_1^{(n-1)}(t) + \cdots + M_n\phi_n^{(n-1)}(t). \end{aligned}$$

Now we evaluate this system at $t = 0$ to obtain

$$\begin{array}{rcll} I & = & M_1\phi_1(0) & + \cdots + M_n\phi_n(0) \\ A & = & M_1\phi_1'(0) & + \cdots + M_n\phi_n'(0) \\ \cdots & \cdots & \cdots & \cdots \\ A^{n-1} & = & M_1\phi_1^{(n-1)}(0) & + \cdots + M_n\phi_n^{(n-1)}(0). \end{array} \qquad (1)$$

Let

$$W = \begin{bmatrix} \phi_1(0) & \ldots & \phi_n(0) \\ \vdots & \ddots & \vdots \\ \phi_1^{(n-1)}(0) & \ldots & \phi_n^{(n-1)}(0) \end{bmatrix}$$

Then W is a nonsingular $n \times n$ matrix; its determinant is just the Wronskian evaluated at 0. So W has an inverse. The above system of equations can now be written:

$$\begin{bmatrix} I \\ A \\ \vdots \\ A^{n-1} \end{bmatrix} = W \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}.$$

Therefore,

$$W^{-1}\begin{bmatrix} I \\ A \\ \vdots \\ A^{n-1} \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}.$$

Having solved for $M_1, \ldots, M_n$ we obtain e^{At} from (‡). □

Remark 6.5.3. Note that this last equation implies that each matrix M_i is a polynomial in the matrix A since W^{-1} is a constant matrix. Specifically, $M_i = p_i(A)$ where

$$p_i(s) = \mathrm{Row}_i(W^{-1})\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix}.$$

Example 6.5.4. Solve $\boldsymbol{y}' = A\boldsymbol{y}$ with initial condition $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, where $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

► **Solution.** The characteristic polynomial is $p(s) = (s-1)^2$. Thus there is only one eigenvalue $r = 1$ with multiplicity 2 so only case $(*_{\text{real}})$ occurs and all of the entries $h_{ij}(t)$ from e^{At} are linear combinations of e^t, te^t. That is $\phi_1(t) = e^t$ while $\phi_2(t) = te^t$. Therefore, Equation $(**)$ is

$$e^{At} = Me^t + Nte^t.$$

Differentiating we obtain

$$\begin{aligned} Ae^{At} &= \quad Me^t + N(e^t + te^t) \\ &= \quad (M+N)e^t + Nte^t. \end{aligned}$$

Now, evaluate each equation at $t = 0$ to obtain:

$$\begin{aligned} I &= \quad M \\ A &= \quad M + N. \end{aligned}$$

Solving for M and N we get

$$\begin{aligned} M &= \quad I \\ N &= \quad A - I. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= Ie^t + (A-I)te^t \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^t + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} te^t \\ &= \begin{bmatrix} e^t + te^t & -te^t \\ te^t & e^t - te^t \end{bmatrix} \end{aligned}$$

We now obtain

$$\begin{aligned} \boldsymbol{y}(t) &= e^{At}\boldsymbol{y}_0 = e^{At} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} e^t + te^t & -te^t \\ te^t & e^t - te^t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} e^t - te^t \\ -te^t + 2e^t \end{bmatrix}. \end{aligned}$$

◀

Example 6.5.5. Compute e^{At} where $A = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & 1 & -1 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$ using Fulmer's method.

▶ **Solution.** The characteristic polynomial is $p(s) = (s-1)(s^2-2s+2)$. The eigenvalues of A are thus $r = 1$ and $r = 1 \pm i$. From $(*_{\text{real}})$ and $(*_{\text{Imag.}})$ each entry of e^{At} is a linear combination of

$$\phi_1(t) = e^t, \quad \phi_2(t) = e^t \sin t, \quad \text{and} \quad \phi_3(t) = e^t \cos t.$$

Therefore

$$e^{At} = Me^t + Ne^t \sin t + Pe^t \cos t.$$

Differentiating twice and simplifying we get the system:

$$\begin{aligned} e^{At} &= Me^t + Ne^t \sin t + Pe^t \cos t \\ Ae^{At} &= Me^t + (N-P)e^t \sin t + (N+P)e^t \cos t \\ A^2e^{At} &= Me^t - 2Pe^t \sin t + 2Ne^t \cos t. \end{aligned}$$

Now evaluating at $t = 0$ gives

$$\begin{aligned} I &= M + P \\ A &= M + N + P \\ A^2 &= M + 2N. \end{aligned}$$

Solving gives

$$\begin{aligned} N &= A - I \\ M &= A^2 - 2A + 2I \\ P &= -A^2 + 2A - I. \end{aligned}$$

Since $A^2 = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 2 & 0 & -2 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$, it follows that

$$N = \begin{bmatrix} 0 & \frac{-1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \quad M = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Hence,

$$\begin{aligned} e^{At} &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} e^t + \begin{bmatrix} 0 & \frac{-1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} e^t \sin t + \begin{bmatrix} \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} \end{bmatrix} e^t \cos t \\ &= \frac{1}{2} \begin{bmatrix} e^t + e^t \cos t & -e^t \sin t & e^t - e^t \cos t \\ 2e^t \sin t & 2e^t \cos t & -2e^t \sin t \\ e^t - e^t \cos t & e^t \sin t & e^t + e^t \cos t \end{bmatrix}. \end{aligned}$$

◀

The technique of this section is convenient for giving an explicit formula for the matrix exponential e^{At} when A is either a 2×2 or 3×3 matrix.

e^A for 2×2 matrices.

Suppose that A is a 2×2 real matrix with characteristic polynomial $p(s) = \det(sI - A) = s^2 + as + b$. We distinguish three cases.

1. $p(s) = (s - r_1)(s - r_2)$ with $r_1 \neq r_2$.

   Then the basic functions are $\phi_1(t) = e^{t_1 t}$ and $\phi_2(t) = e^{r_2 t}$ so that $e^{At} = Me^{r_1 t} + Ne^{r_2 t}$. Equation (1) is then

$$\begin{aligned} I &= M + N \\ A &= r_1 M + r_2 N \end{aligned}$$

which are easily solved to give

$$M = \frac{(A - r_2 I)}{r_1 - r_2} \quad \text{and} \quad N = \frac{(A - r_1 I)}{r_2 - r_1}.$$

Hence, if $p(s)$ has distinct roots, then

$$\boxed{e^{At} = \frac{(A - r_2 I)}{r_1 - r_2} e^{r_1 t} + \frac{(A - r_1 I)}{r_2 - r_1} e^{r_2 t}.} \tag{2}$$

2. $p(s) = (s - r)^2$.

   In this case the basic functions are e^{rt} and te^{rt} so that

   $$(*) \qquad e^{At} = Me^{rt} + Nte^{rt}.$$

   This time it is more convenient to work directly from $(*)$ rather than Equation (1). Multiplying $(*)$ by e^{-rt} and observing that $e^{At}e^{-rt} = e^{At}e^{-rtI} = e^{(A-rI)t}$ (because A commutes with rI), we get

   $$\begin{aligned} M + Nt &= e^{(A-rI)t} \\ &= I + (A - rI)t + \frac{1}{2}(A - rI)^2 t^2 + \cdots . \end{aligned}$$

   Comparing coefficients of t on both sides of the equation we conclude that

   $$M = I, \quad N = (A - rI) \quad \text{and } (A - rI)^n = 0 \text{ for all } n \geq 2.$$

   Hence, if $p(s)$ has a single root of multiplicity 2, then

   $$\boxed{e^{At} = (I + (A - rI)t)\, e^{rt}.} \tag{3}$$

3. $p(s) = (s - \alpha)^2 + \beta^2$ where $\beta \neq 0$, i.e., $p(s)$ has a pair of complex conjugate roots $\alpha \pm \beta$.

   In this case the basic functions are $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$ so that

   $$e^{At} = Me^{\alpha t} \cos \beta t + Me^{\alpha t} \sin \beta t.$$

   Equation (1) is easily checked to be

   $$\begin{aligned} I &= M \\ A &= \alpha M + \beta N. \end{aligned}$$

Solving for M and N then gives

$$\boxed{e^{At} = Ie^{\alpha t}\cos\beta t + \frac{(A-\alpha I)}{\beta}e^{\alpha t}\sin\beta t.} \tag{4}$$

e^A for 3×3 matrices.

Suppose that A is a 3×3 real matrix with characteristic polynomial $p(s) = \det(sI - A)$. As for 2×2 matrices, we distinguish three cases.

1. $p(s) = (s-r_1)(s-r_2)(s-r_3)$ with r_1. r_2, and r_3 distinct roots of $p(s)$.

   This is similar to the first case done above. The basic functions are $e^{r_1 t}$, $e^{r_2 t}$, and $e^{r_3 t}$ so that
   $$e^{At} = Me^{r_1 t} + Ne^{r_2 t} + Pe^{r_3 t}$$
   and the system of equations (1) is
   $$\begin{array}{rcl} I & = & M + N + P \\ A & = & r_1 M + r_2 N + r_3 P \\ A^2 & = & r_1^2 M + r_2^2 N + r_3^2 P. \end{array} \tag{5}$$
   We will use a very convenient trick for solving this system of equations. Suppose that $q(s) = s^2 + as + b$ is any quadratic polynomial. Then in system (5), multiply the first equation by b, the second equation by a, and then add the three resulting equations together. You will get
   $$A^2 + aA + bI = q(A) = q(r_1)M + q(r_2)N + q(r_3)P.$$
   Suppose that we can choose $q(s)$ so that $q(r_2) = 0$ and $q(r_3) = 0$. Since a quadratic can only have 2 roots, we will have $q(r_1) \neq 0$ and hence
   $$M = \frac{q(A)}{q(r_1)}.$$
   But it is easy to find the required $q(s)$, namely, use $q(s) = (s-r_2)(s-r_3)$. This polynomial certainly has roots r_2 and r_3. Thus, we find
   $$M = \frac{(A-r_2 I)(A-r_3 I)}{(r_1-r_2)(r_1-r_3)}.$$

Similarly, we can find N by using $q(s) = (s - r_1)(s - r_3)$ and P by using $q(s) = (s - r_1)(s - r_2)$. Hence, we find the following expression for e^{At}:

$$\boxed{e^{At} = \frac{(A - r_2 I)(A - r_3 I)}{(r_1 - r_2)(r_1 - r_3)} e^{r_1 t} + \frac{(A - r_1 I)(A - r_3 I)}{(r_2 - r_1)(r_2 - r_3)} e^{r_2 t} + \frac{(A - r_1 I)(A - r_2 I)}{(r_3 - r_1)(r_3 - r_2)} e^{r_3 t}.} \tag{6}$$

2. $p(s) = (s - r)^3$, i.e, there is a single eigenvalue of multiplicity 3.

   In this case the basic functions are e^{rt}, te^{rt}, and $t^2 e^{rt}$ so that

$$e^{At} = Me^{rt} + Nte^{rt} + Pt^2 e^{rt}.$$

   As for the case of 2×2 matrices, multiply by e^{-rt} to get

$$\begin{aligned} M + Nt + Pt^2 &= e^{At} e^{-rt} = e^{(A - rI)t} \\ &= I + (A - rI)t + \frac{1}{2}(A - rI)^2 t^2 + \frac{1}{3!}(A - rI)^3 t^3 + \cdots . \end{aligned}$$

   Comparing powers of t on both sides of the equation gives

$$M = I, \quad N = (A - rI), \quad P = \tfrac{(A - rI)^2}{2} \quad \text{and } (A - rI)^n = 0 \text{ if } n \geq 3.$$

   Hence,

$$\boxed{e^{At} = \left(I + (A - rI)t + \frac{1}{2}(A - rI)^2 t^2 \right) e^{rt}.} \tag{7}$$

3. $p(s) = (s - r_1)^2 (s - r_2)$ where $r_1 \neq r_2$. That is, A has one eigenvalue with multiplicity 2 and another with multiplicity 1.

   The derivation is similar to that of the case $p(s) = (s - r)^3$. We will simply record the result:

$$\boxed{e^{At} = \left(I - \frac{(A - r_1 I)^2}{(r_2 - r_1)^2} \right) e^{r_1 t} + \left((A - r_1 I) - \frac{(A - r_1 I)^2}{r_2 - r_1} \right) te^{r_1 t} + \frac{(A - r_1 I)^2}{(r_2 - r_1)^2} e^{r_2 t}.} \tag{8}$$

6.6 Nonhomogeneous Linear Systems

This section will be concerned with the *nonhomogeneous* linear equation

$$(*) \qquad \boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{q}(t),$$

where $A(t)$ and $\boldsymbol{q}(t)$ are matrix functions defined on an interval J in $\mathbb{R}$. The strategy will be analogous to that of Section 3.6 in that *we will assume that we have a fundamental matrix* $\boldsymbol{\Phi}(t) = \begin{bmatrix}\boldsymbol{\varphi}_1(t) & \cdots & \boldsymbol{\varphi}_n(t)\end{bmatrix}$ *of solutions of the associated homogeneous system*

$$(*_{\mathrm{h}}) \qquad \boldsymbol{y}' = A(t)\boldsymbol{y}$$

and we will then use this fundamental matrix $\boldsymbol{\Phi}(t)$ to find a solution $\boldsymbol{y}_p(t)$ of $(*)$ by the method of variation of parameters. Suppose that $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ are two solutions of the nonhomogeneous system $(*)$. Then

$$(\boldsymbol{y}_1-\boldsymbol{y}_2)'(t) = \boldsymbol{y}_1'(t)-\boldsymbol{y}_2'(t) = (A(t)\boldsymbol{y}_1(t)+\boldsymbol{q}(t))-(A(t)\boldsymbol{y}_2(t)+\boldsymbol{q}(t)) = A(t)(\boldsymbol{y}_1(t)-\boldsymbol{y}_2(t))$$

so that $\boldsymbol{y}_1(t)-\boldsymbol{y}_2(t)$ is a solution of the associated homogeneous system $(*_{\mathrm{h}})$. Since $\boldsymbol{\Phi}(t)$ is a fundamental matrix of $(*_{\mathrm{h}})$, this means that

$$\boldsymbol{y}_1(t) - \boldsymbol{y}_2(t) = \boldsymbol{\Phi}(t)\boldsymbol{c} = c_1\boldsymbol{\varphi}_1(t) + \cdots + c_n\boldsymbol{\varphi}_n(t)$$

for some *constant* matrix

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Thus it follows that if we can find one solution, which we will call $\boldsymbol{y}_p(t)$, then all other solutions are determined by the equation

$$\boldsymbol{y}(t) = \boldsymbol{y}_p(t) + \boldsymbol{\Phi}(t)\boldsymbol{c} = \boldsymbol{y}_p(t) + \boldsymbol{y}_h(t)$$

where $\boldsymbol{y}_h(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}$ ($\boldsymbol{c}$ an arbitrary constant vector) is the solution of the associated homogeneous equation $(*_{\mathrm{h}})$. This is frequently expressed by the mnemonic:

$$\boxed{\boldsymbol{y}_{\mathrm{gen}}(t) = \boldsymbol{y}_p(t) + \boldsymbol{y}_h(t),} \qquad (1)$$

or in words: *The general solution of a nonhomogeneous equation is the sum of a particular solution and the general solution of the associated homogeneous equation.* The strategy for finding a particular solution of $(*)$, assuming that we already know $\boldsymbol{y}_h(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}$, is to replace the constant vector $\boldsymbol{c}$ with an unknown vector function

$$\boldsymbol{v}(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix}.$$

That is, we will try to choose $\boldsymbol{v}(t)$ so that the vector function

$$(\dagger) \qquad \boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\boldsymbol{v}(t) = v_1(t)\boldsymbol{\varphi}_1(t) + \cdots + v_n(t)\varphi_n(t)$$

is a solution of $(*)$. Differentiating $\boldsymbol{y}(t)$ gives $\boldsymbol{y}'(t) = \boldsymbol{\Phi}'(t)\boldsymbol{v}(t) + \boldsymbol{\Phi}(t)\boldsymbol{v}'(t)$, and substituting this expression for $\boldsymbol{y}'(t)$ into $(*)$ gives

$$\boldsymbol{\Phi}'(t)\boldsymbol{v}(t) + \boldsymbol{\Phi}(t)\boldsymbol{v}'(t) = \boldsymbol{y}'(t) = A(t)\boldsymbol{y}(t) + \boldsymbol{q}(t) = A(t)\boldsymbol{\Phi}(t)\boldsymbol{v}(t) + \boldsymbol{q}(t).$$

But $\boldsymbol{\Phi}'(t) = A(t)\boldsymbol{\Phi}(t)$ (since $\boldsymbol{\Phi}(t)$ is a fundamental matrix for $(*_{\mathrm{h}})$) so $\boldsymbol{\Phi}'(t)\boldsymbol{v}(t) = A(t)\boldsymbol{\Phi}(t)\boldsymbol{v}(t)$ cancels from both sides of the equation to give

$$\boldsymbol{\Phi}(t)\boldsymbol{v}'(t) = \boldsymbol{q}(t).$$

Since $\boldsymbol{\Phi}(t)$ is a fundamental matrix Theorem 6.3.4 implies that $\boldsymbol{\Phi}(t)^{-1}$ exists, and we arrive at an equation

$$(\ddagger) \qquad \boldsymbol{v}'(t) = \boldsymbol{\Phi}(t)^{-1}\boldsymbol{q}(t)$$

for $\boldsymbol{v}'(t)$. Given an initial point $t_0 \in J$, we can then integrate $(\ddagger)$ to get

$$\boldsymbol{v}(t) - \boldsymbol{v}(t_0) = \int_{t_0}^{t} \boldsymbol{\Phi}(u)^{-1}\boldsymbol{q}(u)\,du,$$

and multiplying by $\boldsymbol{\Phi}(t)$ gives

$$\boldsymbol{y}(t) - \boldsymbol{\Phi}(t)\boldsymbol{v}(t_0) = \boldsymbol{\Phi}(t)\int_{t_0}^{t} \boldsymbol{\Phi}(u)^{-1}\boldsymbol{q}(u)\,du.$$

But if $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$, then $\boldsymbol{y}_0 = \boldsymbol{y}(t_0) = \boldsymbol{\Phi}(t_0)\boldsymbol{v}(t_0)$, and hence $\boldsymbol{v}(t_0) = \boldsymbol{\Phi}(t_0)^{-1}\boldsymbol{y}_0$. Substituting this expression in the above equation, we arrive at the following result, which we formally record as a theorem.

Theorem 6.6.1. *Suppose that $A(t)$ and $\boldsymbol{q}(t)$ are continuous on an interval J and $t_0 \in J$. If $\boldsymbol{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\boldsymbol{y}' = A(t)\boldsymbol{y}$ then the unique solution of the nonhomogeneous initial value problem*

$$\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{q}(t), \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

is

$$\boxed{\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\left(\boldsymbol{\Phi}(t_0)\right)^{-1}\boldsymbol{y}_0 + \boldsymbol{\Phi}(t)\int_{t_0}^{t} \boldsymbol{\Phi}(u)^{-1}\boldsymbol{q}(u)\,du.} \qquad (2)$$

Remark 6.6.2. The procedure described above is known as **variation of parameters for nonhomogeneous systems**. It is completely analogous to the technique of variation of parameters previously studied for a single second order linear nonhomogeneous differential equation. See Section 3.6.

Remark 6.6.3. How does the solution of $\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{q}(t)$ expressed by Equation (2) correlate to the general mnemonic expressed in Equation (1)? If we let the initial condition $\boldsymbol{y}_0$ vary over all possible vectors in $\mathbb{R}^n$, then $\boldsymbol{y}_h(t)$ is the first part of the expression on the right of Equation (2). That is $\boldsymbol{y}_h(t) = \boldsymbol{\Phi}(t)\left(\boldsymbol{\Phi}(t_0)\right)^{-1}\boldsymbol{y}_0$. The second part of the expression on the right of Equation (2) is the particular solution of $\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{q}(t)$ corresponding to the specific initial condition $\boldsymbol{y}(t_0) = \boldsymbol{0}$. Thus, in the language of (1)

$$\boldsymbol{y}_p = \boldsymbol{\Phi}(t)\int_{t_0}^{t} \boldsymbol{\Phi}(u)^{-1}\boldsymbol{q}(u)\,du.$$

Finally, $\boldsymbol{y}_{\text{gen}}(t)$ is just the function $\boldsymbol{y}(t)$, and the fact that it is the *general solution* is just the observation that the initial vector $\boldsymbol{y}_0$ is allowed to be arbitrary.

Example 6.6.4. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ -e^t \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{3}$$

▶ **Solution.** From Example 6.3.5 we have that

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix}$$

is a fundamental matrix for the homogeneous system $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$, which is the associated homogeneous system $\boldsymbol{y}' = A\boldsymbol{y}$ for the nonhomogeneous system (3). Then $\det \boldsymbol{\Phi}(t) = -3e^t$ and

$$\boldsymbol{\Phi}(t)^{-1} = \frac{1}{-3e^t}\begin{bmatrix} -e^{-t} & -e^{-t} \\ -2e^{2t} & e^{2t} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} e^{-2t} & e^{-2t} \\ 2e^{t} & -e^{t} \end{bmatrix}.$$

Then

$$\boldsymbol{\Phi}(0)^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and hence

$$\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(0)^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}.$$

which is the first part of $\boldsymbol{y}(t)$ in Equation (2). Now compute the second half of Equation (2):

$$\begin{aligned}
\boldsymbol{\Phi}(t)\int_{t_0}^{t}\boldsymbol{\Phi}(u)^{-1}\boldsymbol{q}(u)\,du &= \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t}\end{bmatrix}\int_0^t \frac{1}{3}\begin{bmatrix} e^{-2u} & e^{-2u} \\ 2e^{u} & -e^{u}\end{bmatrix}\begin{bmatrix} 0 \\ -e^{u}\end{bmatrix} du \\
&= \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t}\end{bmatrix}\int_0^t \frac{1}{3}\begin{bmatrix} -e^{-u} \\ e^{2u}\end{bmatrix} du \\
&= \begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t}\end{bmatrix}\begin{bmatrix} \frac{1}{3}(e^{-t}-1) \\ \\ \frac{1}{6}(e^{2t}-1)\end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3}(e^{t}-e^{2t})+\frac{1}{6}(e^{t}-e^{-t}) \\ \\ \frac{2}{3}(e^{t}-e^{2t})-\frac{1}{6}(e^{t}-e^{-t})\end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}e^{t}-\frac{1}{3}e^{2t}-\frac{1}{6}e^{-t} \\ \\ \frac{1}{2}e^{t}-\frac{2}{3}e^{2t}+\frac{1}{6}e^{-t}\end{bmatrix}.
\end{aligned}$$

Putting together the two parts which make up $\boldsymbol{y}(t)$ in Equation (2) we get

$$\boldsymbol{y}(t) = \begin{bmatrix} e^{-t} \\ -e^{-t}\end{bmatrix} + \begin{bmatrix} \frac{1}{2}e^{t}-\frac{1}{3}e^{2t}-\frac{1}{6}e^{-t} \\ \\ \frac{1}{2}e^{t}-\frac{2}{3}e^{2t}+\frac{1}{6}e^{-t}\end{bmatrix}.$$

We will leave it as an exercise to check our work by substituting the above expression for $\boldsymbol{y}(t)$ back into the system (3) to see that we have in fact found the solution. ◀

If the linear system $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}(t)$ is constant coefficient, then a fundamental matrix for the associated homogeneous system is $\boldsymbol{\Phi}(t) = e^{At}$. Since $(e^{At})^{-1} = e^{-At}$, it follows that $e^{At}(e^{At_0})^{-1} = e^{A(t-t_0)}$ and hence Theorem 6.6.1 has the following form in this situation.

Theorem 6.6.5. *Suppose that A is a constant matrix and $\boldsymbol{q}(t)$ is a continuous vector function on an interval J and $t_0 \in J$. Then the unique solution of the nonhomogeneous initial value problem*

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}(t), \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

is

$$\boxed{\boldsymbol{y}(t) = e^{A(t-t_0)}\boldsymbol{y}_0 + e^{At}\int_{t_0}^{t} e^{-Au}\boldsymbol{q}(u)\,du.} \tag{4}$$

You should compare the statement of this theorem with the solution of the first order linear initial value problem as expressed in Corollary 1.3.9.

Example 6.6.6. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} e^t \\ 2e^t \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

▶ **Solution.** In this example, $A = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$, $\boldsymbol{q}(t) = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix}$, $t_0 = 0$, and $\boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since the characteristic polynomial of A is $p(s) = \det(sI - A) = (s-1)^2$, the fundamental matrix e^{At} can be computed from Equation (3):

$$\begin{aligned} e^{At} &= (I + (A - I)t)e^t \\ &= \begin{bmatrix} 1-2t & t \\ -4t & 1+2t \end{bmatrix} e^t. \end{aligned}$$

Since $e^{-At} = e^{A\cdot(-t)}$, we can compute e^{-At} by simply replacing t by $-t$ in the formula for e^{At}:

$$e^{-At} = \begin{bmatrix} 1+2t & -t \\ 4t & 1-2t \end{bmatrix} e^{-t}.$$

Then applying Equation (4) give

$$\begin{aligned} \boldsymbol{y}(t) &= e^{At}\boldsymbol{y}_0 + e^{At}\int_0^t e^{-Au}\boldsymbol{q}(u)\,du \\ &= \begin{bmatrix} 1-2t & t \\ -4t & 1+2t \end{bmatrix} e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1-2t & t \\ -4t & 1+2t \end{bmatrix} e^t \int_0^t \begin{bmatrix} 1+2u & -u \\ 4u & 1-2u \end{bmatrix} e^{-u} \begin{bmatrix} e^u \\ 2e^u \end{bmatrix} du \\ &= \begin{bmatrix} (1-2t)e^t \\ -4te^t \end{bmatrix} + \begin{bmatrix} 1-2t & t \\ -4t & 1+2t \end{bmatrix} e^t \int_0^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} du \\ &= \begin{bmatrix} (1-2t)e^t \\ -4te^t \end{bmatrix} + \begin{bmatrix} 1-2t & t \\ -4t & 1+2t \end{bmatrix} e^t \begin{bmatrix} t \\ 2t \end{bmatrix} \\ &= \begin{bmatrix} (1-2t)e^t \\ -4te^t \end{bmatrix} + \begin{bmatrix} te^t \\ 2te^t \end{bmatrix}. \end{aligned}$$

◀

If we take the initial point $t_0 = 0$ in Theorem 6.6.5, then we can get a further refinement of Equation (4) by observing that

$$e^{At}\int_0^t e^{-Au}\boldsymbol{q}(u)\,du = \int_0^t e^{A(t-u)}\boldsymbol{q}(u)\,du = e^{At} * \boldsymbol{q}(t)$$

where $e^{At} * \boldsymbol{q}(t)$ means the matrix of functions obtained by a formal matrix multiplication in which ordinary product of entries are replaced by the convolution product. For example, if $B(t) = \begin{bmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{bmatrix}$ and $\boldsymbol{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$, then by $B(t) * \boldsymbol{q}(t)$ we mean the matrix

$$B(t) * \boldsymbol{q}(t) = \begin{bmatrix} (h_{11} * q_1)(t) + (h_{12} * q_2)(t) \\ \\ (h_{21} * q_1)(t) + (h_{22} * q_2)(t) \end{bmatrix} = \begin{bmatrix} \int_0^t h_{11}(t-u)q_1(u)\,du + \int_0^t h_{12}(t-u)q_2(u)\,du \\ \\ \int_0^t h_{21}(t-u)q_1(u)\,du + \int_0^t h_{22}(t-u)q_2(u)\,du \end{bmatrix}.$$

With this observation we can give the following formulation of Theorem 6.6.5 in terms of the convolution product.

Theorem 6.6.7. *Suppose that A is a constant matrix and $\boldsymbol{q}(t)$ is a continuous vector function on an interval J and $0 \in J$. Then the unique solution of the nonhomogeneous initial value problem*

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}(t), \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$$

is

$$\boxed{\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0 + e^{At} * \boldsymbol{q}(t).} \tag{5}$$

Remark 6.6.8. For low dimensional examples, the utility of this result is greatly enhanced by the use of the explicit formulas (2) – (8) from the previous section and the table of convolution products (Table C.3).

Example 6.6.9. Solve the following constant coefficient non-homogeneous linear system:

$$\begin{aligned} y_1' &= y_2 + e^{3t} \\ y_2' &= 4y_1 + e^t. \end{aligned} \tag{6}$$

► **Solution.** In this system the coefficient matrix is $A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$ and $\mathbf{q}(t) = \begin{bmatrix} e^{3t} \\ e^t \end{bmatrix}$. The associated homogeneous equation $\boldsymbol{y}' = A\boldsymbol{y}$ has already been studied in Example 6.4.1 where we found that a fundamental matrix is

$$\boldsymbol{z}(t) = \begin{bmatrix} \frac{1}{2}(e^{2t} + e^{-2t}) & \frac{1}{4}(e^{2t} - e^{-2t}) \\ \\ e^{2t} - e^{-2t} & \frac{1}{2}(e^{2t} + e^{-2t}) \end{bmatrix}. \tag{7}$$

Since, $\boldsymbol{z}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it follows that $\boldsymbol{z}(t) = e^{At}$. If we write e^{At} in terms of its columns, so that $e^{At} = \begin{bmatrix} \boldsymbol{z}_1(t) & \boldsymbol{z}_2(t) \end{bmatrix}$ then we conclude that a solution of the initial value problem

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}(t), \quad \boldsymbol{y}(0) = \mathbf{0}$$

is given by

$$\begin{aligned}
\boldsymbol{y}_p(t) &= e^{At} * \boldsymbol{q}(t) \\
&= \boldsymbol{z}_1(t) * q_1(t) + \boldsymbol{z}_2(t) * q_2(t) \\
&= \mathbf{z}_1 * e^{3t} + \mathbf{z}_2 * e^t \\
&= \begin{bmatrix} \frac{1}{2}\left(e^{2t} + e^{-2t}\right) \\ e^{2t} - e^{-2t} \end{bmatrix} * e^{3t} + \begin{bmatrix} \frac{1}{4}\left(e^{2t} - e^{-2t}\right) \\ \frac{1}{2}\left(e^{2t} + e^{-2t}\right) \end{bmatrix} * e^t \\
&= \begin{bmatrix} \frac{1}{2}\left(e^{3t} - e^{2t} + \frac{1}{5}\left(e^{3t} - e^{-2t}\right)\right) \\ e^{3t} - e^{2t} - \frac{1}{5}\left(e^{3t} - e^{-2t}\right) \end{bmatrix} + \begin{bmatrix} \frac{1}{4}\left(e^{2t} - e^t - \frac{1}{3}\left(e^t - e^{-2t}\right)\right) \\ \frac{1}{2}\left(e^{2t} - e^t + \frac{1}{3}\left(e^{3t} - e^{-2t}\right)\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{5}e^{3t} - \frac{1}{2}e^{2t} - \frac{1}{10}e^{-2t} \\ \frac{4}{5}e^{3t} - e^{2t} + \frac{1}{5}e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{4}e^{2t} + \frac{1}{12}e^{-2t} - \frac{1}{3}e^t \\ \frac{1}{2}e^{2t} - \frac{1}{6}e^{-2t} - \frac{1}{3}e^t \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{5}e^{3t} - \frac{1}{3}e^t - \frac{1}{4}e^{2t} - \frac{1}{60}e^{-2t} \\ \frac{4}{5}e^{3t} - \frac{1}{3}e^t - \frac{1}{2}e^{2t} + \frac{1}{30}e^{-2t} \end{bmatrix}.
\end{aligned}$$

The general solution to (6) is then obtained by taking $\boldsymbol{y}_p(t)$ and adding to it the general solution $\boldsymbol{y}_h(t) = e^{At}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ of the associated homogeneous equation. Hence,

$$y_{\text{gen}} = \begin{bmatrix} \frac{1}{2}c_1 + \frac{1}{4}c_2 \\ c_1 + \frac{1}{2}c_2 \end{bmatrix} e^{2t} + \begin{bmatrix} \frac{1}{2}c_1 - \frac{1}{4}c_2 \\ -c_1 + \frac{1}{2}c_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{3}{5}e^{3t} - \frac{1}{3}e^t - \frac{1}{4}e^{2t} - \frac{1}{60}e^{-2t} \\ \frac{4}{5}e^{3t} - \frac{1}{3}e^t - \frac{1}{2}e^{2t} + \frac{1}{30}e^{-2t} \end{bmatrix}.$$

◀

Exercises

In part (a) of each exercise in Section 4.4, you were asked to find e^{At} for the given matrix A. Using your answer to that exercise, solve the nonhomogeneous equation

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}(t), \quad \boldsymbol{y}(0) = \mathbf{0},$$

where A is the matrix in the corresponding exercise in Section 4.4 and $\boldsymbol{q}(t)$ is the following matrix function. (*Hint:* Theorem 4.6.5 and Example 4.6.7 should prove particularly useful to study for these exercises.)

1. $\boldsymbol{q}(t) = \begin{bmatrix} e^{-t} \\ 2e^{t} \end{bmatrix}$
2. $\boldsymbol{q}(t) = \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$
3. $\boldsymbol{q}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$
5. $\boldsymbol{q}(t) = \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix}$
7. $\boldsymbol{q}(t) = \begin{bmatrix} 0 \\ \sin t \end{bmatrix}$
11. $\boldsymbol{q}(t) = \begin{bmatrix} e^{t} \\ e^{2t} \\ e^{-t} \end{bmatrix}$

Appendix A

COMPLEX NUMBERS

A.1 Complex Numbers

The history of numbers starts in the stone age, about 30,000 years ago. Long before humans could read or write, a caveman who counted the deer he killed by a series of notches carved into a bone, introduced mankind to the natural counting numbers $1, 2, 3, 4, \cdots$. To be able to describe quantities and their relations among each other, the first human civilizations expanded the number system first to rational numbers (integers and fractions) and then to real numbers (rational numbers and irrational numbers like $\sqrt{2}$ and π). Finally in 1545, to be able to tackle more advanced computational problems in his book about *The Great Art (Ars Magna)*, Girolamo Cardano brought the complex numbers (real numbers and "imaginary" numbers like $\sqrt{-1}$) into existence. Unfortunately, 450 years later and after changing the whole of mathematics forever, complex numbers are still greeted by the general public with suspicion and confusion.

The problem is that most folks still think of numbers as entities that are used solely to describe quantities. This works reasonably well if one restricts the number universe to the real numbers, but fails miserably if one considers complex numbers: no one will ever catch $\sqrt{-1}$ pounds of crawfish, not even a mathematician.

In mathematics, numbers are used to do computations, and it is a matter of fact that nowadays almost all serious computations in mathematics require somewhere along the line the use of the largest possible number system given to mankind: the complex numbers. Although complex numbers are useless to describe the weight of your catch of the day, they are indispensable if, for example, you want to make a sound mathematical prediction about the behavior of any biological, chemical, or physical system in time.

Since the ancient Greeks, the algebraic concept of a real number is associated with the geometric concept of a point on a line (the *number line*), and these two concepts are still used as synonyms. Similarly, complex numbers can be given a simple, concrete, geometric interpretation as points in a plane; i.e., any **complex number** z corresponds to a point in the plane (the *number plane*) and can be represented in Cartesian coordinates as $z = (x, y)$, where x and y are real numbers.

We know from Calculus II that every point $z = (x, y)$ in the plane can be described also in **polar coordinates** as $z = [\alpha, r]$, where $r = |z| = \sqrt{x^2 + y^2}$ denotes the **radius (length, modulus, norm, absolute value, distance to the origin)** of the point z, and where $\alpha = \arg(z)$ is the angle (in radians) between the positive x-axis and the line joining 0 and z. Note that α can be determined by the equation $\tan \alpha = y/x$, when $x \neq 0$, and knowledge of which quadrant the number z is in. Be aware that α is not unique; adding $2\pi k$ to α gives another angle (argument) for z.

We identify the real numbers with the x-axis in the plane; i.e., a real number x is identified with the point $(x, 0)$ of the plane, and vice versa. Thus, the real numbers are a subset of the complex numbers. As pointed out above, in mathematics the defining property of numbers is not that they describe quantities, but that we can do computations with them; i.e., we should be able to add and multiply them. The addition and multiplication of points in the plane are defined in such a way that

(a) they coincide on the x-axis (real numbers) with the usual addition and multiplication of real numbers, and

(b) all rules of algebra for real numbers (points on the x-axis) extend to complex numbers (points in the plane).

Addition: we add complex numbers coordinate-wise in Cartesian coordinates. That is, if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2).$$

Multiplication: we multiply complex numbers in polar coordinates by adding their angles α and multiplying their radii r (in polar coordinates). That is, if $z_1 = [\alpha_1, r_1]$ and $z_2 = [\alpha_2, r_2]$, then

$$z_1 z_2 := [\alpha_1 + \alpha_2, r_1 r_2].$$

The definition of multiplication of points in the plane is an extension of the familiar rule for multiplication of signed real numbers: plus times plus is plus, minus times minus is plus, plus times minus is minus. To see this, we identify the real numbers 2 and -3 with the complex numbers $z_1 = (2,0) = [0,2]$ and $z_2 = (-3,0) = [\pi, 3]$. Then

$$z_1 z_2 = [0+\pi, 2\cdot 3] = [\pi, 6] = (-6,0) = -6$$
$$z_2^2 = [\pi,3][\pi,3] = [\pi+\pi, 3\cdot 3] = [2\pi, 9] = (0,9) = 9,$$

which is not at all surprising since we all know that $2\cdot -3 = -6$, and $(-3)^2 = 9$. What this illustrates is part (a); namely, the arithmetic of real numbers is the same whether considered in their own right, or considered as a subset of the complex numbers.

To demonstrate the multiplication of complex numbers (points in the plane) which are not real (not on the x-axis), consider $z_1 = (1,1) = [\frac{\pi}{4}, \sqrt{2}]$ and $z_2 = (1,-1) = [-\frac{\pi}{4}, \sqrt{2}]$. Then

$$z_1 z_2 = [\frac{\pi}{4} - \frac{\pi}{4}, \sqrt{2}\cdot\sqrt{2}] = [0,2] = (2,0) = 2.$$

If one defines multiplication of points in the plane as above, the point $i := (0,1) = [\frac{\pi}{2}, 1]$ has the property that

$$i^2 = [\frac{\pi}{2} + \frac{\pi}{2}, 1\cdot 1] = [\pi, 1] = (-1,0) = -1.$$

Thus, one defines

$$\sqrt{-1} := i = (0,1).$$

Notice that $\sqrt{-1}$ is not on the x-axis and is therefore not a real number. Employing i and identifying the point $(1,0)$ with the real number 1, one can now write a complex number $z = (x,y)$ in the **standard algebraic form** $z = x + iy$; i.e.,

$$z = (x,y) = (x,0) + (0,y) = x(1,0) + (0,1)y = x + iy.$$

If $z = (x,y) = x + iy$, then the real number $x := \operatorname{Re} z$ is called the **real part** and the real number $y := \operatorname{Im} z$ is called the **imaginary part** of z (which is one of the worst misnomers in the history of science since there is absolutely nothing imaginary about y).

The basic rules of algebra carry over to complex numbers if we simply remember the identity $i^2 = -1$. In particular, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + iy_1x_2 + x_1iy_2 + iy_1iy_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \end{aligned}$$

This algebraic rule is often easier to use than the geometric definition of multiplication given above. For example, if $z_1 = (1,1) = 1 + i$ and $z_2 = (1,-1) = 1 - i$, then the computation $z_1 z_2 = (1+i)(1-i) = 1 - i^2 = 2$ is more familiar than the one given above using the polar coordinates of z_1 and z_2.

The formula for **division** of two complex numbers (points in the plane) is less obvious, and is most conveniently expressed in terms of the **complex conjugate** $\overline{z} := (x, -y) = x - iy$ of a complex number $z = (x, y) = x + iy$. Note that $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$, and

$$|z|^2 = x^2 + y^2 = z\overline{z}, \qquad \operatorname{Re} z = \frac{z + \overline{z}}{2} \quad \text{and} \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

Using complex conjugates, we divide complex numbers using the formula

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{\overline{w}}{\overline{w}} = \frac{z\overline{w}}{|w|^2}.$$

As an example we divide the complex number $z = (1, 1) = 1+i$ by $w = (3, -1) = 3-i$. Then

$$\frac{z}{w} = \frac{1+i}{3-i} = \frac{(1+i)(3+i)}{(3-i)(3+i)} = \frac{2+4i}{10} = \frac{1}{5} + \frac{2}{5}i = (\frac{1}{5}, \frac{2}{5}).$$

Let $z = (x, y)$ be a complex number with polar coordinates $z = [\alpha, r]$. Then $|z| = r = \sqrt{x^2 + y^2}$, $\operatorname{Re} z = x = |z| \cos\alpha$, $\operatorname{Im} z = y = |z| \sin\alpha$, and $\tan\alpha = y/x$. Thus we obtain the following **exponential form** of the complex number z; i.e.,

$$z = [\alpha, r] = (x, y) = |z|(\cos\alpha, \sin\alpha) = |z|(\cos\alpha + i\sin\alpha) = |z|e^{i\alpha},$$

where the last identity requires Euler's formula relating the complex exponential and trigonometric functions. The most natural means of understanding the validity of Euler's formula is via the power series expansions of e^x, $\sin x$, and $\cos x$, which were studied in calculus. Recall that the exponential function e^x has a power series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which converges for all $x \in \mathbb{R}$. This infinite series makes perfectly good sense if x is replaced by *any* complex number z, and moreover, it can be shown that the resulting series converges for all $z \in \mathbb{C}$. Thus, we *define* the **complex exponential function** by means of the convergent series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{1}$$

It can be shown that this function e^z satisfies the expected functional equation, that is

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}.$$

Since $e^0 = 1$, it follows that $\dfrac{1}{e^z} = e^{-z}$. Euler's formula will be obtained by taking $z = it$ in Definition 1; i.e.,

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \cdots \\ &= (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots) + i(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots) = \cos t + i \sin t = (\cos t, \sin t), \end{aligned}$$

where one has to know that the two series following the last equality are the Taylor series expansions for $\cos t$ and $\sin t$, respectively. Thus we have proved Euler's formula, which we formally state as a theorem.

Theorem A.1.1 (Euler's Formula). *For all $t \in \mathbb{R}$ we have*

$$e^{it} = \cos t + i \sin t = (\cos t, \sin t) = [t, 1].$$ □

Example A.1.2. Write $z = -1 + i$ in exponential form.

► **Solution.** Note that $z = (-1, 1)$ so that $x = -1$, $y = 1$, $r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, and $\tan \alpha = y/x = -1$. Thus, $\alpha = \frac{3\pi}{4}$ or $\alpha = \frac{7\pi}{4}$. But z is in the 2nd quadrant, so $\alpha = \frac{3\pi}{4}$. Thus the polar coordinates of z are $[\frac{3\pi}{4}, \sqrt{2}]$ and the exponential form of z is $\sqrt{2}e^{i\frac{3\pi}{4}}$. ◄

Example A.1.3. Write $z = 2e^{\frac{\pi i}{6}}$ in Cartesian form.

► **Solution.**

$$z = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i = (\sqrt{3}, 1).$$

◄

Using the exponential form of a complex number gives yet another description of the multiplication of two complex numbers. Suppose that z_1 and z_2 are given in exponential form, that is, $z_1 = r_1 e^{i\alpha_1}$ and $z_2 = r_2 e^{i\alpha_2}$. Then

$$z_1 z_2 = (r_1 e^{i\alpha_1})(r_2 e^{i\alpha_2}) = (r_1 r_2) e^{i(\alpha_1 + \alpha_2)}.$$

Of course, this is nothing more than a reiteration of the definition of multiplication of complex numbers; i.e., if $z_1 = [\alpha_1, r_1]$ and $z_2 = [\alpha_2, r_2]$, then $z_1 z_2 := [\alpha_1 + \alpha_2, r_1 r_2]$.

Example A.1.4. Find $z = \sqrt{i}$. That is, find all z such that $z^2 = i$.

▶ **Solution.** Observe that $i = (0,1) = [\pi/2, 1] = e^{\frac{\pi}{2}i}$. Hence, if $z = e^{i\frac{\pi}{4}}$ then $z^2 = (e^{i\frac{\pi}{4}})^2 = e^{i\frac{\pi}{2}} = i$ so that

$$z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(1+i).$$

Also note that $i = e^{(\frac{\pi}{2}+2\pi)i}$ so that $w = e^{(\frac{\pi}{4}+\pi)i} = e^{\frac{\pi}{4}}e^{\pi i} = -e^{\frac{\pi}{4}} = -z$ is another square root of i. ◀

Example A.1.5. Find all complex solutions to the equation $z^3 = 1$.

▶ **Solution.** Note that $1 = e^{2\pi ki}$ for any integer k. Thus the cube roots of 1 are obtained by dividing the possible arguments of 1 by 3 since raising a complex number to the third power multiplies the argument by 3 (and also cubes the modulus). Thus the possible cube roots of 1 are 1, $\omega = e^{\frac{2\pi}{3}i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = e^{\frac{4\pi}{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}$. ◀

We will conclude this section by summarizing some of the properties of the complex exponential function. The proofs are straight forward calculations based on Euler's formula and are left to the reader.

Theorem A.1.6. *Let $z = x + iy$. Then*

1. $e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$. *That is* $\operatorname{Re} e^z = e^x \cos y$ *and* $\operatorname{Im} e^z = e^x \sin y$.
2. $|e^z| = e^x$. *That is, the modulus of e^z is the exponential of the real part of z.*
3. $\cos y = \dfrac{e^{iy} + e^{-iy}}{2}$
4. $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$ □

Example A.1.7. Compute the real and imaginary parts of the complex function

$$z(t) = (2+3i)e^{i\frac{5t}{2}}.$$

▶ **Solution.** Since $z(t) = (2+3i)(\cos\frac{5t}{2} + i\sin\frac{5t}{2}) = (2\cos\frac{5t}{2} - 3\sin\frac{5t}{2}) + (3\cos\frac{5t}{2} + 2\sin\frac{5t}{2})i$, it follows that $\operatorname{Re} z(t) = 2\cos\frac{5t}{2} - 3\sin\frac{5t}{2}$ and $\operatorname{Im} z(t) = 3\cos\frac{5t}{2} + 2\sin\frac{5t}{2}$. ◀

Exercises

1. Let $z = (1,1)$ and $w = (-1,1)$. Find $z \cdot w$, $\frac{z}{w}$, $\frac{w}{z}$, z^2, $\sqrt{z}$ and z^{11} using
   (a) the polar coordinates,
   (b) the standard forms $x + iy$,
   (c) the exponential forms.

2. Find
   (a) $(1+2i)(3+4i)$ (b) $(1+2i)^2$ (c) $\dfrac{1}{2+3i}$ (d) $\dfrac{1}{(2-3i)(2+4i)}$ (e) $\dfrac{4-2i}{2+i}$.

3. Solve each of the following equations for z and check your result.
   (a) $(2+3i)z + 2 = i$ (b) $\dfrac{z-1}{z-i} = \dfrac{2}{3}$ (c) $\dfrac{2+i}{z} + 1 = 2 + i$ (d) $e^z = -1$.

4. Find the modulus of each of the following complex numbers.
   (a) $4 + 3i$ (b) $(2+i)^2$ (c) $\dfrac{13}{5+12i}$ (d) $\dfrac{1+2it-t^2}{1+t^2}$ where $t \in \mathbb{R}$.

5. Find all complex numbers z such that $|z-1| = |z-2|$. What does this equation mean geometrically?

6. Determine the region in the complex plane $\mathbb{C}$ described by the inequality
   $$|z-1| + |z-3| < 4.$$
   Give a geometric description of the region.

7. Compute: (a) $\sqrt{2+2i}$ (b) $\sqrt{3+4i}$

8. Write each of the following complex numbers in exponential form.
   (a) $3+4i$ (b) $3-4i$ (c) $(3+4i)^2$ (d) $\dfrac{1}{3+4i}$ (e) -5 (f) $3i$

9. Find the real and imaginary parts of each of the following functions.
   (a) $(2+3i)e^{(-1+i)t}$ (b) $ie^{2it+\pi}$ (c) $e^{(2+3i)t}e^{(-3-i)t}$

10. (a) Find the value of the sum
    $$1 + e^z + e^{2z} + \cdots + e^{(n-1)z}.$$
    Hint: Compare the sum to a finite geometric series.

(b) Compute $\sin(\frac{2\pi}{n}) + \sin(\frac{4\pi}{n}) + \cdots + \sin(\frac{(n-1)\pi}{n})$

11. Find all of the cube roots of $8i$. That is, find all solutions to the equation $z^3 = 8i$.

12. By multiplying out $e^{i\theta}e^{i\phi}$ and comparing it to $e^{i(\theta+\phi)}$, rederive the addition formulas for the cosine and sine functions.

Appendix B

SELECTED ANSWERS

Chapter 1

Section 1.1

1. 1
2. 2
3. 1
4. 2
5. 2
6. $y_3(t)$
7. $y_1(t)$, $y_4(t)$
8. $y_1(t)$, $y_2(t)$, $y_3(t)$
9. $y_2(t)$, $y_3(t)$.

16. $y(t) = \frac{1}{2}e^{2t} - t + c$
17. $y(t) = -e^{-t}(t+1) + c$
18. $y(t) = t + \ln|t| + c$
19. $y(t) = \frac{t^3}{3} + \frac{t^2}{2} + c_1 t + c_2$

20. $y(t) = -\frac{2}{3}\sin 3t + c_1 t + c_2$

22. $y(t) = 3e^{-t} + 3t - 3$

23. $y(t) = 1/(1 + e^t)$

24. $y(t) = -18(t + 1)^{-1}$

25. $y(t) = \frac{1}{2}e^{2t} - t + \frac{7}{2}$

26. $y(t) = -e^{-t}(t + 1)$

27. $y(t) = -\frac{2}{3}\sin 3t + t + 1$

28. $R' = kR$ where k is a proportionality constant.

29. $y' = k(1 - y)$, $y(0) = 1$ where k is a proportionality constant.

30. $P' = kP$ where k is a proportionality constant.

31. $P' = kP(M - P)$ where k is a proportionality constant.

32. $T' = k(32 - T)$, $T(0) = 70$ where k is a proportionality constant.

33. 900 ft at 5 sec; 15.8 seconds to hit the ground.

Section 1.2

1. separable

2. not separable

3. separable

4. not separable

5. separable

6. not separable

7. separable

8. not separable

9. separable

12. $y^4 = 2t^2 + c$

13. $2y^5 = 5(t+2)^2 + c$

14. $y(t^2 + c) = -2$, $y = 0$

15. $y = \dfrac{-3}{t^3 + c}$, $y = 0$

16. $y = 1 - c\cos t$, $y = 1$

17. $y^{1-n} = \dfrac{1-n}{1+m} t^{m+1} + c$, $y = 0$

18. $y = \dfrac{4ce^{4t}}{1 - ce^{4t}}$, $y = 4$

19. $y^2 + 1 = ce^{2t}$

20. $y = \tan(t + c)$

21. $t^2 + y^2 + 2\ln|t| = c$

22. $\tan^{-1} t + y - 2\ln|y + 1| = c$, $y = -1$

23. $y^2 = e^t + c$

24. $y \ln|c(1 - t)| = 1$

25. $ce^t = y(t + 2)^2$

26. $y = 0$

27. $y = 0$

28. $y = x^2 e^x$

29. $y = 4e^{-t^2}$

30. $y = \sec^{-1}(\sqrt{2t^2})$

31. $y = 2\sqrt{u^2 + 1}$

32. 121.7°

34. 52.6°

36. 205°

Section 1.3

3. $y(t) = te^{2t} + 4e^{2t}$

4. $y(t) = -\dfrac{1}{4}e^{-2t} + \dfrac{17}{4}e^{2t}$

5. $y(t) = \dfrac{1}{t}e^{t} - \dfrac{e}{t}$

6. $y(t) = \dfrac{1}{2t}[e^{2t} - e^2]$

7. $y(t) = \frac{t + \sin t \cos t}{2\cos t} + c \sec t$

8. $y(t) = e^{-t^2/2} \int_0^t e^{s^2/2}\, ds + e^{-t^2/2}$

9. $y(t) = \dfrac{t \ln t}{m+1} - \dfrac{t}{(m+1)^2} + ct^{-m}$

10. $y(t) = \frac{\sin(t^2) + C}{t}$

11. $y(t) = \frac{1}{t+1}(-2 + ct)$

12. $y(t) = b/a + ce^{-at}$

13. $y(t) = 1$

14. $y(t) = t(t+1)^2 + c(t+1)^2$

15. $y(t) = \left(-\frac{1}{2t^2} - \frac{1}{t}\right) t^2 - \frac{3}{2}t^2$

16. $y(t) = te^{-at} + ce^{-at}$

17. $y(t) = \frac{1}{a+b}e^{bt} + ce^{-at}$

18. $y(t) = \frac{t^{n+1}}{n+1}e^{-at} + ce^{-at}$

19. $y(t) = \frac{t+c}{\cos t}$

20. $y = 2 + ce^{-(\ln t)^2}$

21. $y(t) = t^n e^t + ct^n$

22. $y(t) = (t-1)e^{2t} + (a+1)e^t$

23. $y(t) = \frac{t^2}{5} + \frac{9}{5}t^{-3}$

24. $y(t) = \frac{1}{t}\left[1 + \frac{2(2a-1)}{t}\right]$

25. $y(t) = (10-t) - 8(1 - \frac{t}{10})^4$. Note that $y(10) = 0$, so the tank is empty after 10 min.

26. (a) $T = 45$ min; (b) $y(t) = \frac{1}{2}(10+2t) - 50(10+2t)^{-1}$ for $0 \le t \le 45$ so $y(45) = 50 - \frac{1}{2} = 49.5$ lb. (c) $\lim_{t\to\infty} y(t) = 50$. Once the tank is full, the inflow and outflow rates will be equal and the brine in the tank will stabilize to the concentration of the incoming brine, i.e., .5 lb/gal. Since the tank holds 100 gal, the total amount present will approach $.5 \times 100 = 50$ lb.

27. If $y(t)$ is the amount of salt present at time t (measured in pounds), then $y(t) = 80e^{-.04t}$, and the concentration $c(t) = .8e^{-.04t}$ lb/gal.

28. (a) Differential equation: $P'(t) + (r/V)P(t) = rc$. If P_0 denotes the initial amount of pollutant in the lake, then $P(t) = Vc + (P_0 - Vc)e^{-(r/V)t}$. The limiting concentration is c.

    (b) (i) $t_{1/2} = (V/r)\ln 2$; (ii) $t_{1/10} = (V/r)\ln 10$

(c) Lake Erie: $t_{1/2} = 1.82$ years, $t_{1/10} = 6.05$ years, Lake Ontario: $t_{1/2} = 5.43$ years, $t_{1/10} = 18.06$ years

29. (a) 10 minutes

    (b) 1600/3 grams

30. $1 - e^{-1}$ grams/liter

Section 1.4

Section 1.5

2. $$\begin{aligned} y_1(t) &= 1 - t + \frac{t^2}{2} \\ y_2(t) &= 1 - t + t^2 - \frac{t^3}{6} \\ y_3(t) &= 1 - t + t^2 - \frac{t^3}{3} + \frac{t^4}{4!} \end{aligned}$$

3. $$\begin{aligned} y_1(t) &= \frac{t^2}{2} \\ y_2(t) &= \frac{t^2}{2} + \frac{t^5}{20} \\ y_3(t) &= \frac{t^2}{2} + \frac{t^5}{20} + \frac{t^8}{160} + \frac{t^{11}}{4400} \end{aligned}$$

4. Unique solution

5. Not guaranteed unique

6. Unique solution

7. Unique solution

8. Not guaranteed unique

9. (a) $y(t) = t + ct^2$

   (b) Every solution satisfies $y(0) = 0$. There is no contradiction to Theorem 1.5.2 since, in normal form, the equation is $y' = \frac{2}{t}y - 1 = F(t, y)$ and $F(t, y)$ is not continuous for $t = 0$.

10. (a) $F(t,y) = y^2$ so both $F(t,y) = y^2$ and $F_y(t,y) = 2y$ are continuous for any (t_0, y_0). Hence Theorem 1.5.2 applies.

    (b) $y(t) = 0$ is defined for all t; $y(t) = \dfrac{1}{1-t}$ is only defined on $(-\infty, 1)$.

11. No. Both $y_1(t)$ and $y_2(t)$ would be solutions to the initial value problem $y' = F(t,y)$, $y(0) = 0$. If $F(t,y)$ and $F_y(t,y)$ are both continuous near $(0,0)$, then the initial value problem would have a unique solution by Theorem 1.5.2.

12. There is no contraction to Theorem 1.5.2 since, in the normal form $y' = \dfrac{3}{t}y = F(t,y)$ has a discontinuous $F(t,y)$ near $(0,0)$.

Section 1.6

2. $ty + y^2 - \dfrac{1}{2}t^2 = c$
3. Not Exact
4. $ty^2 + t^3 = c$
5. Not Exact
6. $t^2y + y^3 = 2$
7. $(y - t^2)^2 - 2t^4 = c$
8. $y = \dfrac{1}{3}t^2 - \dfrac{c}{t}$
9. $y^4 = 4ty + c$
10. $b + c = 0$
11. $y = (1-t)^{-1}$
12. $y^2(tl^2 + 1 + e^{t^2}) = 1$
13. $y = (c\sqrt{1-t^2} - 5)^{-1}$
14. $y^2 = (1 + ce^{t^2})^{-1}$
15. $y^2 = (t + \frac{1}{2} + ce^{2t})^{-1}$
16. $y = -\sqrt{2e^{2t} - e^t}$

17. $y = 2t^2 + ct^{-2}$

18. $y = (1 - \ln t)^{-1}$

19. $y(t) = e^{\frac{1}{2}(t^2-1)} + 1$

20. $t^2y + y^3 = c$

21. $y = (\ln\left|\frac{t}{t+1}\right| + c)^2$

22. $y = c\left|\frac{t-1}{t+3}\right|^{1/4}$

23. $t\sin y + y\sin t + t^2 = c$

24. $y = \frac{t}{t-1}\left(\frac{1}{2}t^2 - 2t + \ln|t| + c\right)$

Chapter 2

Section 2.1

3. $\frac{5}{s-2}$

4. $\frac{3}{s+7} - \frac{42}{s^4}$

5. $\frac{2}{s^3} - \frac{5}{s^2} + \frac{4}{s}$

6. $\frac{6}{s^4} + \frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s}$

7. $\frac{8s+25}{(s+3)(s+4)}$

8. $\frac{s^2+15s+37}{(s+3)(s+4)^2}$

9. $\frac{s+2}{s^2+4}$

10. $\frac{4}{(s-1)((s-1)^2+4)}$

11. $\dfrac{9s+3}{9s^2+6s+55}$

12. $\dfrac{2}{s^3}+\dfrac{2}{(s-2)^2}+\dfrac{1}{s-4}$

13. $\dfrac{\sqrt{2}}{s+(1.1)}+\dfrac{0.123}{(s+(1.1))^2}$

14. $\dfrac{5s-6}{s^2+4}+\dfrac{4}{s}$

15. $\dfrac{8(s-5)+22}{(s-5)^2+4}$

16. $\dfrac{12s^2-16}{(s^2+4)^3}$

17. $\dfrac{b-a}{(s+a)(s+b)}$

18. $\dfrac{s^2+2b^2}{s(s^2+4b^2)}$

19. $\dfrac{2b^2}{s(s^2+4b^2)}$

20. $\dfrac{b}{s^2+4b^2}$

21. $\dfrac{s}{s^2-b^2}$

22. $\dfrac{b}{s^2-b^2}$

24. (a), (c), (e), (g), (i) are functions in class $\mathcal{E}$.

Section 2.2

1. (a) **(R)**; (b) **(PR)**; (c) **(R)**; (d) **(PR)**; (e) **(PR)**; (f) **(NR)**; (g) **(NR)**; (h) **(PR)**; (i) **(NR)**

2. -5

3. $3t-2t^2$

4. $2e^{-3t/2}$
5. $3\cos\sqrt{2}t$
6. $\frac{2}{3}\cos\sqrt{\frac{2}{3}}t$
7. $\frac{2}{\sqrt{3}}\sin\sqrt{3}t$
8. $\cos\sqrt{\frac{2}{3}}t + \sqrt{\frac{2}{3}}\sin\sqrt{\frac{2}{3}}t$
9. te^{-3t}
10. $e^{-3t}(2-11t)$
11. $e^{-3t}(2t - \frac{11}{2}t^2)$
12. $e^{2t}(2t + \frac{3}{2}t^2 - \frac{1}{6}t^3)$
13. $e^{-2t}\cos 3t$
14. $e^{t}\cos 3t$
15. $e^{-3t}(2\cos 3t - \frac{1}{3}\sin 3t)$
16. $e^{-2t}(3\cos\sqrt{2}t - 4\sqrt{2}\sin\sqrt{2}t)$
17. $e^{-t/2}(\frac{5}{2}\cos(t/2) + \frac{1}{2}\sin(t/2))$
18. $3e^{3t} - 2e^{2t}$
19. $\frac{5}{6}(e^{2t} - e^{-4t})$
20. $4e^{5t} - 2e^{t}$

Section 2.3

1. $\frac{1}{7}(e^{-2t} - e^{5t})$
2. $\frac{1}{2}(7e^{t} + 3e^{-3t})$
3. $\frac{1}{8}(13e^{5t} - 5e^{-3t})$
4. $e^{2t} - e^{t}$
5. $\frac{1}{12}(37e^{-7t} + 23e^{5t})$

6. $e^t + 2e^{-t}$

7. $\frac{1}{8}(25e^{7t} - 9e^{-t})$

8. $\frac{1}{2}(9e^t - 30e^{2t} + 25e^{3t})$

9. $\frac{1}{6}((3 + \sqrt{3})e^{\sqrt{3}t} + (3 - \sqrt{3})e^{-\sqrt{3}t})$

10. $\frac{1}{2}(2e^{2t} - e^t + e^{-5t})$

11. $\frac{7}{6}t^3 e^{-4t}$

12. $te^{3t} + \frac{3}{2}t^2 e^{3t}$

13. $e^{-3t} - 5te^{3t} + \frac{3}{2}t^2 e^{-3t}$

14. $18e^{-t} - 13e^{-2t} - 36te^{-2t}$

15. $\frac{1}{54}(5e^{5t} + 3te^{5t} - 5e^{-t} + 21te^{-t})$

16. $\frac{1}{2}e^{-t} \sin 4t$

17. $2e^{-t} \cos 4t - \frac{1}{2}e^{-t} \sin 4t$

18. $\frac{5}{2}e^{-3t/2}$

19. $-\frac{3}{16}e^{-3t/2} + \frac{7}{16}e^{t/2}$

20. $3e^{2t} \cos \sqrt{3}t + \frac{8}{\sqrt{3}}e^{2t} \sin \sqrt{3}t$

21. $3e^{-3t} \cos 2t - \frac{7}{2}e^{-3t} \sin 2t$

22. $2e^{-2t} \cos 5t - \frac{1}{5}e^{-2t} \sin 5t$

23. $2e^t - 2 \cos t + \sin t$

24. $2e^{-t} + \cos 2t - \sin 2t$

25. $2e^t - 2e^{-t} \sin 2t$

26. $\cos 2t + \frac{15}{16} \sin 2t - \frac{5}{4}t \sin 2t + \frac{9}{8}t \cos 2t$

Section 2.4

1. $y(t) = \frac{8}{9}e^{-6t} + \frac{1}{9}e^{3t}$
2. $y(t) = 2e^{4t}$
3. $y(t) = \frac{-3}{4} + \frac{11}{4}e^{4t}$
4. $y(t) = \frac{1}{16}(-1 + 33e^{4t} - 4t)$
5. $y(t) = \frac{-20}{9}e^{-9t} + \frac{1}{9}(2 - 18t + 81t^2))$
6. $y(t) = \frac{1}{10}(3e^{3t} - 3\cos t + \sin t)$
7. $y(t) = \frac{1}{2}t^2e^{-2t}$
8. $y(t) = 6e^{3t} - 5\cos t - 15\sin t$
9. $y(t) = 2 + \frac{1}{2}\sin 2t$
10. $y(t) = 2 - 3e^t + 3e^{2t}$
11. $y(t) = -7e^t + 4e^{2t} - te^t$
12. $y(t) = \frac{1}{10}e^t - \frac{1}{26}e^{-3t} - \frac{4}{65}\cos 2t - \frac{7}{65}\sin 2t$
13. $y(t) = -3\cos t + 4\sin t + (3 + 7t)e^{-3t}$
14. $y(t) = \cos 5t - \frac{1}{5}\sin 5t$
15. $y(t) = \left(\frac{1}{2} + 4t\right)e^{-4t}$
16. $y(t) = (2t^2 - 2t - 1)e^{2t}$
17. $y(t) = \frac{2}{\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$
18. $y(t) = e^t - 1 - \dfrac{t^2}{2} - \dfrac{t^3}{6}$
19. $y(t) = \dfrac{1}{2}(e^t - 2(1 + t) + \cos t + \sin t)$
20. $y(t) = (e^t + e^{-t} + 2\cos t)/4$
21. $y(t) = e^t + t^3$
22. $y(t) = 0$

23. $y(t) = e^{-3t} - 3e^{-t} + 2$

24. $y(t) = \frac{1}{20}(e^t + e^{-t} - 2\cos 3t)$

25. $y(t) = t\sin t - t^2 \cos t$

Section 2.5

1. $\dfrac{t^3}{6}$

2. $\dfrac{t^5}{20}$

3. $3 - 3\cos t$

4. $\dfrac{7e^{4t} - 12t - 7}{16}$

5. $\dfrac{2e^{2t} - 2\cos 2t - 3\sin 2t}{13}$

6. $\dfrac{1}{2}(1 - \cos 2t + \sin 2t)$

7. $\dfrac{1}{108}(1 - 6t + 18t^2 - e^{-6t})$

8. $\dfrac{1}{3}(-\sin t + 2\sin 2t)$

9. $\dfrac{1}{6}(e^{2t} - e^{-4t})$

10. $\dfrac{t^{n+2}}{(n+1)(n+2)}$

11. $\dfrac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$

12. $\dfrac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$

13. $\begin{cases} \dfrac{b\sin at - a\sin bt}{b^2 - a^2} & \text{if } b \neq a \\ \dfrac{\sin at - at\cos at}{2a} & \text{if } b = a \end{cases}$

14. $\begin{cases} \dfrac{a\cos at - a\cos bt}{b^2 - a^2} & \text{if } b \neq a \\ \dfrac{1}{2} t \sin at & \text{if } b = a \end{cases}$

15. $\begin{cases} \dfrac{a\sin at - b\sin bt}{a^2 - b^2} & \text{if } b \neq a \\ \dfrac{1}{2a}(at\cos at + \sin at) & \text{if } b = a \end{cases}$

17. $F(s) = \dfrac{4}{s^3(s^2+4)}$

18. $F(s) = \dfrac{6}{s^4(s+3)}$

19. $F(s) = \dfrac{6}{s^4(s+3)}$

20. $F(s) = \dfrac{s}{(s^2+25)(s-4)}$

21. $F(s) = \dfrac{2s}{(s^2+4)(s^2+1)}$

22. $F(s) = \dfrac{4}{(s^2+4)^2}$

23. $\dfrac{1}{6}(e^{2t} - e^{-4t})$

24. $\dfrac{1}{4}(-e^{t} + e^{5t})$

25. $\dfrac{1}{2}(\sin t - t\cos t)$

26. $\dfrac{1}{2} t \sin t$

27. $\dfrac{1}{216}(-e^{-6t} + 1 - 6t + 18t^2)$

28. $\dfrac{1}{13}(2e^{3t} - 2\cos 2t - 3\sin 2t)$

29. $\dfrac{1}{17}(4e^{4t} - 4\cos t + \sin t)$

30. $\dfrac{e^{at} - e^{bt}}{a - b}$

31. $\dfrac{at - \sin at}{a^3}$

32. $\displaystyle\int_0^t g(\tau)e^{-2(t-\tau)}\,d\tau$

33. $\displaystyle\int_0^t g(\tau)\cos\sqrt{2}(t-\tau)\,d\tau$

34. $\displaystyle\frac{1}{\sqrt{3}}\int_0^t \sin\sqrt{3}(t-\tau)\,f(\tau)\,d\tau$

35. $\displaystyle\int_0^t (t-\tau)e^{-2(t-\tau)}f(\tau)\,d\tau$

36. $\displaystyle\int_0^t e^{-(t-\tau)}\sin 2(t-\tau)\,f(\tau)\,d\tau$

37. $\displaystyle\int_0^t \left(e^{-2(t-\tau)} - e^{-3(t-\tau)}\right) f(\tau)\,d\tau$

Chapter 3

Section 3.1

	linear	constant coefficient	homogeneous/nonhomogeneous
(1)	no		
(2)	yes	yes	homogeneous
(3)	yes	yes	nonhomogeneous
(4)	no		
(5)	yes	yes	nonhomogeneous
(6)	yes	yes	nonhomogeneous
(7)	no		
(8)	yes	yes	nonhomogeneous
(9)	yes	no	homogeneous
(10)	no		
(11)	yes	no	homogeneous
(12)	yes	no	homogeneous

	$\boldsymbol{L}(1)$	$\boldsymbol{L}(t)$	$\boldsymbol{L}(e^{-t})$	$\boldsymbol{L}(\cos 2t)$
(13)	1	t	$2e^{-t}$	$\cos 2t$
(14)	1	1	$(t+1)e^{-t}$	$(-4t+1)\cos 2t$
(15)	-3	$1-3t$	$-2e^{-t}$	$-11\cos 2t - 2\sin 2t$
(16)	5	$5t+6$	0	$\cos 2t - 12\sin 2t$
(17)	-4	$-4t$	$-3e^{-t}$	$-8\cos 2t$
(18)	-1	0	$(t^2-t-1)e^{-t}$	$(-4t^2-1)\cos 2t - 2t\sin 2t$

19. $\boldsymbol{L}(e^{rt}) = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2+br+c)e^{rt}$.

20. $C = \frac{-3}{4}$

21. $C_1 = \frac{-3}{4}$ and $C_2 = \frac{1}{2}$

22. no

23. yes, $C = 1$.

25. (c) i. $y = \dfrac{1}{2}e^t + 2e^{2t} - \dfrac{3}{2}e^{3t}$

(c) ii. $y = \dfrac{1}{2}e^t - 2e^{2t} + \dfrac{3}{2}e^{3t}$

(c) iii. $y = \dfrac{1}{2}e^t - 7e^{2t} + \dfrac{11}{2}e^{3t}$

(c) iv. $y = \dfrac{1}{2}e^t + (-1+3a-b)e^{2t} + \left(\dfrac{1}{2} - 2a + b\right)e^{3t}$

26. (c) i. $y = \dfrac{1}{6}t^5 + \dfrac{10}{3}t^2 - \dfrac{5}{2}t^3$

(c) ii. $y = \dfrac{1}{6}t^5 - \dfrac{2}{3}t^2 + \dfrac{1}{2}t^3$

(c) iii. $y = \dfrac{1}{6}t^5 - \dfrac{17}{3}t^2 + \dfrac{9}{2}t^3$

(c) iv. $y = \dfrac{1}{6}t^5 + \left(\dfrac{1}{3} + 3a - b\right)t^2 + \left(-\dfrac{1}{2} - 2a + b\right)t^3$

28. Maximal intervals are $(-\infty, -1)$, $(-1,\ 1)$, $(1,\ \infty)$

29. $(k\pi,\ (k+1)\pi)$ where $k \in \mathbb{Z}$

30. $(-\infty,\ \infty)$

31. $(3,\ \infty)$

32. $(-\infty,\ -2)$, $(-2,\ 0)$, $(0\,2)$, $(2,\ \infty)$

33. Theorem 5.2.1 doesn't apply since if the initial value problem is put in standard form, then $a(t) = -\frac{4}{t}$ and $b(t) = \frac{6}{t^2}$ are not continuous at $t = 0$, so the theorem says nothing about initial value problems which start at $t_0 = 0$.

34. $\varphi(t_0) = \varphi'(t_0) = 0$ so that φ and 0 are both solutions of the initial value problem

$$y'' + a(t)y' + b(t)y = 0, \quad y(t_0) = 0,\ y'(t_0) = 0.$$

Hence $\varphi = 0$ by Theorem 5.2.1.

Section 3.2

1. dependent
2. independent
3. independent
4. dependent
5. independent
6. dependent
7. dependent
8. dependent
9. (a) Note that $\frac{\varphi_1(t)}{\varphi_2(t)} = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases}$. Therefore φ_1 is not a multiple of φ_2.
   (b) Check separately the cases $t > 0$, $t < 0$, and $t = 0$.
   (c) Theorem 3.2.6 only applies to pairs of functions which are solutions of a standard second order linear differential equation on an interval I, in this case, $I = \mathbb{R}$. The conclusion is that φ_1 and φ_2 are not solutions of such a differential equation.
   (d) Simply substitute into the equation.
   (e) When the given equation is put in standard form, the coefficient of y' is $-\frac{2}{t}$, which is not continuous on $\mathbb{R}$, so that Theorem 5.3.10 does not apply.

Section 3.3

1. $\{e^{t},\ e^{-2t}\}$
2. $\{e^{-4t},\ e^{4t}\}$
3. $\{e^{-3t},\ 1\}$
4. $\left\{e^{\frac{-t}{2}},\ e^{3t}\right\}$
5. $\left\{e^{\sqrt{2}t},\ e^{-\sqrt{2}t}\right\}$
6. $\left\{e^{(1+\sqrt{2})t},\ e^{(1-\sqrt{2})t}\right\}$
7. $\{e^{3t},\ te^{3t}\}$
8. $\{e^{-2t},\ te^{-2t}\}$
9. $\{1,\ t\}$
10. $\left\{e^{\frac{3t}{2}},\ te^{\frac{3t}{2}}\right\}$
11. $\{\sin t,\ \cos t\}$
12. $\left\{\sin\frac{t}{\sqrt{5}},\ \cos\frac{t}{\sqrt{5}}\right\}$
13. $\{e^{2t}\sin 3t,\ e^{2t}\cos 3t\}$
14. $\{e^{-t}\sin t,\ e^{-t}\cos t\}$
15. $\{e^{4t}\sin t,\ e^{4t}\cos t\}$
16. $\left\{e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right),\ e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right\}$
17. *Solution:* The characteristic polynomial is $s^2 - s - 6 = (s-3)(s+2)$ and thus has distinct real roots 3 and -2. The general solution is $y = ae^{3t} + be^{-2t}$. Differentiating gives $y' = 3ae^{3t} - 2be^{-2t}$. The initial conditions imply

$$\begin{aligned} a + b &= \ 2 \\ 3a - 2b &= \ 1. \end{aligned}$$

The solution is $a = 1$ and $b = 1$. Thus the solution to the initial value problem is

$$y = e^{3t} + e^{-2t}.$$

18. $y = te^t$

19. $y = 5e^{-t} - 2e^{-3t}$

20. $y = 2\cos 2t - \sin 2t$

21. $y = \sqrt{7}e^{\sqrt{7}t} - \sqrt{7}e^{-\sqrt{7}t}$

22. $y = e^{-t}\cos t + e^{-t}\sin t$

23. *Solution:* This function is a linear combination of the two functions e^t and e^{-3t}, which form a fundamental set for the constant coefficient equation with characteristic polynomial

$$p(s) = (s-1)(s+3) = s^2 + 2s - 3.$$

The homogeneous equation with this characteristic polynomial is

$$\boxed{y'' + 2y' - 3y = 0.}$$

There is no lower order equation which will work, since such an equation would have the form $y' + ay = 0$ for some $a \in \mathbb{R}$, and all of the solutions of this equation are of the form $y = Ce^{at}$ where C is a constant. The given function $e^t + 2e^{-3t}$ is not a pure exponential function so it is not possible to choose C and a so that $e^t + 2e^{-3t} = Ce^{at}$.

24. $y'' + 9y' + 14 = 0$

25. $y'' + 4y' + 4 = 0$

26. $y'' + 25y = 0$

27. $y'' - 4y' + 13y = 0$

28. $y'' + 2y' + y = 0$

29. Not a solution of a constant coefficient second order homogeneous equation since it is not a linear combination of any of the functions listed in Theorem 3.3.1.

30. *Solution:* Since the characteristic polynomial of this equation is $p(s) = s^2 + 5s + 6 = (s+3)(s+2)$, it follows that the general solution of this equation is

$$y(t) = c_1e^{-2t} + c_2e^{-3t}$$

and since both exponentials have negative exponents, it follows that $\lim_{t\to\infty} y(t) = 0$, no matter what c_1 and c_2 are.

Section 3.4

1. $c_1 t + c_2 t^{-2}$
2. $c_1 t^{1/2} + c_2 t^3$
3. $c_1 t^{\sqrt{2}} + c_2 t^{-\sqrt{2}}$
4. $c_1 t^{1/2} + c_2 t^{1/2} \ln t$
5. $c_1 t^{-3} + c_2 t^{-3} \ln t$
6. $c_1 t^2 + c_2 t^{-2}$
7. $c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)$
8. $c_1 t^2 \cos(3 \ln t) + c_2 t^2 \sin(3 \ln t)$
9. $y = \dfrac{1}{3}(t - t^{-2})$
10. $y = 2t^{1/2} - t^{1/2} \ln t$
11. $y = -3 \cos(2 \ln t) + 2 \sin(2 \ln t)$
12. No solution is possible.

Section 3.5

1. $y = c_1 e^{-2t} + c_2 e^{-t} + 2$
2. $y = c_1 e^{-2t} + c_2 e^{-t} + 2e^t$
3. $y = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{10}(\sin t - 3 \cos t)$
4. $y = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{10}(3 \sin t + \cos t)$
5. $y = c_1 e^{-2t} + c_2 e^{-t} + 4 + e^t + \frac{1}{5}(\sin t - 3 \cos t)$
6. $y = c_1 e^{4t} + c_2 e^{-t} - e^t$
7. $y = c_1 e^{4t} + c_2 e^{-t} + te^{4t}$
8. $y = c_1 e^t + c_2 e^{3t} + 2 \cos t - 4 \sin t$
9. $y = c_1 e^t + c_2 e^{3t} + \cos t$

10. $y = c_1 e^{2t} + c_2 e^{-2t} + 2t3^{2t} + 3$

11. $y = c_1 e^t + c_2 e^{2t} + t^3$

12. $y = c_1 e^t + c_2 e^{2t} + t^2 + 3t + 4$

13. $y = c_1 \sin 2t + c_2 \cos 2t + e^t - t$

14. $y = c_1 \sin 2t + c_2 \cos 2t + e^t - t^2 + \frac{1}{2}$

15. $y = e^{-t/2}\left(c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t\right) + t^2 - 2t$

16. $y = c_1 e^{4t} + c_2 e^{-2t} - te^t - 2e^{-t}$

17. $y = c_1 + c_2 e^{3t} - \frac{1}{5}e^{2t}(\cos t + 3\sin t)$

18. $y = c_1 + c_2 e^{-t} + \frac{t^3}{3}$

19. $y = c_1 + c_2 e^{-t} + \frac{t^2}{2} - t - \frac{1}{10}(2\sin 2t + \cos 2t)$

20. $y = c_1 \cos t + c_2 \sin t + \frac{1}{2}t \sin t$

21. $y = c_1 \cos t + c_2 \sin t - t(t\cos t - \sin t)$

22. $y = c_1 e^{4t} + c_2 e^{-t} + 3 - 4t + 4\cos 2t + 3\sin 2t$

23. $y = c_1 e^{-t} + c_2 e^{-4t} + e^{2t} + \frac{1}{2}te^{-t}$

24. $y = c_1 e^{2t} + c_2 e^{-t} + \frac{3}{2} - 3t - 2te^{-t}$

25. $y = c_1 \cos t + c_2 \cos t + \frac{1}{2} + \frac{1}{6}\cos 2t$

26. $y = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{2}t^2 e^{2t}$

27. $y = \frac{10}{21}e^{6t} + \frac{45}{28}e^{-t} - \frac{1}{12}e^{3t}$

28. $y = e^{-t}(2 + 4\sin 2t - 2\cos 2t)$

29. $y = 2e^{2t} - 2\cos t - 4\sin t$

30. $y = e^{2t} - \frac{1}{2}e^{-2t} + 2t - \frac{1}{2}$

31. $y = \frac{1}{3}e^{2t} + \frac{1}{6}e^{-t} - \frac{3}{2}\sin t + \frac{1}{2}\cos t.$

32. $y = \cos t + \frac{2}{3}\cos 3t + \sin 3t$

33. $y = e^{2t} + te^t$

34. $y = -\frac{5}{3}e^{2t} + \frac{5}{2}e^t + \frac{1}{6}e^{-t}$

Section 3.6

1. $y = a\sin t + b\cos t - \cos t \ln(|\sec t + \tan t|)$
2. $y = \frac{-t\cos t}{2} + \frac{\sin t}{4} + a\sin t + b\cos t$
3. $y = ae^{2t} + be^{-2t} + (\frac{t}{4} - \frac{1}{16})e^{2t}$
4. $y = ae^t + bte^t + (-1 + \ln t)te^t$
5. $y = \frac{1}{2}e^{3t} + ae^t + be^{2t}$
6. $y = \frac{1}{4}e^t + ae^t\cos 2t + be^t\sin 2t$
7. $y = -\cos t \ln \sec t + t\sin t + a\sin t + b\cos t$
8. $\frac{1}{9}(-3te^{-3t} - e^{-3t}) + a + be^{-3t}$
9. $\frac{t^4}{6} + at + bt^2$
10. $y = t^3 + t + a + bt$

Section 3.7

1. $y(t) = \sqrt{8}\cos\left(5t - \dfrac{3\pi}{4}\right)$
2. $y(t) = 5\cos(2t + .9273)$
3. $y(t) = \dfrac{\sqrt{5}}{2}\cos(4t - .4634)$
4. $y(t) = 2\cos\left(t - \dfrac{4\pi}{3}\right)$
5. underdamped
6. critically damped
7. overdamped
8. overdamped
9. underdamped
10. critically damped

11. $\sqrt{2}e^{-t}\cos\left(2t+\dfrac{\pi}{4}\right)$

12. $2e^{-2t}\cos\left(t-\dfrac{\pi}{6}\right)$

13. $5e^{-0.2t}\cos(5t+.6435)$

14. $t=\pi/8$ for problem 11, $t=2\pi/3$ for problem 12, $t=.1855$ for problem 13

15.

16.

17. $-2\sin t\sin 8t$

18. $2\sin\dfrac{1}{2}t\sin\dfrac{19}{2}t$

Section 3.8

1. $y=\frac{1}{2}\sin 4t$, Maximum displacement is $\frac{1}{2}$ feet.
2. $y=\frac{\sqrt{12}}{6}\cos(\sqrt{12}t)e^{-2t}$. (Underdamped) Maximum displacement is .273 feet.
3. $y=\frac{1}{2}te^{-4t}$. (Critically damped) Maximum displacement is .184 feet.
4. $y=\frac{1}{3}e^{-2t}-\frac{1}{3}e^{-8t}$. (Overdamped) Maximum displacement is .0104 feet.

Chapter 4

Section 4.1

1. (c)
2. (g)
3. (e)
4. (a)
5. (f)
6. (d)

7. (h)

8. (b)

9. $\frac{-22}{3}$

10. $\frac{9}{4}$

11. 4

12. $1 + \ln 2$

13. $\frac{11}{2}$

14. 0

15. 5

16. $\frac{44}{3}$

17. (a) A,B
    (b) A,B,C
    (c) A
    (d) none

18. (a) A,B
    (b) A,C
    (c) A,B,C,D
    (d) A,B,C

19. $y(t) = \begin{cases} \frac{t}{3} - \frac{1}{9} + \frac{e^{-3t}}{9} & \text{if } 0 \leq t < 1 \\ \frac{1}{3} - \frac{e^{-3(t-1)}}{9} + \frac{e^{-3t}}{9} & \text{if } 1 \leq t < \infty \end{cases}$

20. $y(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ -t + e^{t-1} & \text{if } 1 \leq t < 2 \\ t - 2 - 2e^{t-2} + e^{t-1} & \text{if } 2 \leq t < 3 \\ e^{t-3} - 2e^{t-2} + e^{t-1} & \text{if } 3 \leq t < \infty \end{cases}$

21. $y = \begin{cases} \frac{\sin}{2} - \frac{\cos t}{2} - \frac{3e^{-(t-\pi)}}{2} & \text{if } 0 \leq t < \pi \\ -e^{-(t-\pi)} & \text{if } \pi \leq t < \infty \end{cases}$

22. $y(t) = \begin{cases} -t + e^t - e^{-t} & \text{if } 0 \le t < 1 \\ e^t - e^{t-1} - e^{-t} & \text{if } 1 \le t < \infty \end{cases}$

23. $y(t) = \begin{cases} e^{2t} - 2te^{2t} & \text{if } 0 \le t < 2 \\ 1 + e^{2t} - 5e^{2(t-2)} - 2te^{2t} + 2te^{2(t-2)} & \text{if } 2 \le t < \infty \end{cases}$

Section 4.2

9. (a) $(t-2)\chi_{[2,\infty)}$; (b) $(t-2)h(t-2)$; (c) e^{-2s}/s^2.

10. (a) $t\chi_{[2,\infty)}$; (b) $th(t-2)$; (c) $e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$.

11. (a) $(t+2)\chi_{[2,\infty)}$; (b) $(t+2)h(t-2)$; (c) $e^{-2s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$.

12. (a) $(t-4)^2\chi_{[4,\infty)}$; (b) $(t-4)^2h(t-4)$; (c) $e^{-4s}\frac{2}{s^3}$.

13. (a) $t^2\chi_{[4,\infty)}$; (b) $t^2h(t-4)$; (c) $e^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right)$.

14. (a) $(t^2-4)\chi_{[4,\infty)}$; (b) $(t^2-4)h(t-4)$; (c) $e^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{12}{s}\right)$.

15. (a) $(t-4)^2\chi_{[2,\infty)}$; (b) $(t-4)^2h(t-2)$; (c) $e^{-2s}\left(\frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}\right)$.

16. (a) $e^{t-4}\chi_{[4,\infty)}$; (b) $e^{t-4}h(t-4)$; (c) $e^{-4s}\frac{1}{s-1}$.

17. (a) $e^t\chi_{[4,\infty)}$; (b) $e^th(t-4)$; (c) $e^{-4(s-1)}\frac{1}{s-1}$.

18. (a) $e^{t-4}\chi_{[6,\infty)}$; (b) $e^{t-4}h(t-6)$; (c) $e^{-6s+2}\frac{1}{s-1}$.

19. (a) $te^t\chi_{[4,\infty)}$; (b) $te^th(t-4)$; (c) $e^{-4(s-1)}\left(\frac{1}{(s-1)^2} + \frac{4}{s-1}\right)$.

20. (a) $\chi_{[0,4)}(t) - \chi_{[4,5)}(t)$; (b) $1 - 2h_4 + h_5$; (c) $\frac{1}{s} - \frac{2e^{-4s}}{s} + \frac{e^{-5s}}{s}$.

21. (a) $t\chi_{[0,1)}(t) + (2-t)\chi_{[1,2)}(t) + \chi_{[2,\infty)}(t)$; (b) $t - (2-2t)h_1 + (t-1)h_2$; (c) $\frac{1}{s^2} + \frac{2e^{-s}}{s^2} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$.

22. (a) $t\chi_{[0,1)}(t) + (2-t)\chi_{[1,\infty)}(t)$; (b) $t + (2-2t)h_1$; (c) $\frac{1}{s^2} + \frac{2e^{-s}}{s^2}$.

23. (a) $\sum_{n=0}^{\infty}(t-n)\chi_{[n,n+1)}(t)$; (b) $t - \sum_{n=1}^{\infty} h_n$; (c) $\frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}$.

24. (a) $\sum_{n=0}^{\infty}\chi_{[2n,2n+1)}(t)$; (b) $\sum_{n=0}^{\infty}(-1)^n h_n$; (c) $\frac{1}{s(1+e^{-s})}$.

25. (a) $t^2\chi_{[0,4)}(t) + 4\chi_{[2,3)}(t) + (7-t)\chi_{[3,4)}(t)$; (b) $t^2 + (4-t^2)h_2 + (3-t)h_3 + (7-t)h_4$; (c) $\frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} - \frac{4}{s^2}\right) - \frac{e^{-3s}}{s^2} - e^{-4s}\left(\frac{1}{s^2} - \frac{4}{s}\right)$.

26. (a) $\sum_{n=0}^{\infty}(2n+1-t)\chi_{[2n,2n+2)}(t)$; (b) $-(t+1) + 2\sum_{n=0}^{\infty} h_{2n}$; (c) $-\frac{1}{s^2} - \frac{1}{s} + \frac{2}{s(1-e^{-2s})}$.

27. (a) $\chi_{[0,2)}(t)+(3-t)\chi_{[2,3)}(t)+2(t-3)\chi_{[3,4)}(t)+2\chi_{[4,\infty)}(t)$; (b) $1+(2-t)h_2+(3t-9)h_3-(2t-8)h_4$; (c) $\frac{1}{s}-\frac{e^{-2s}}{s^2}+\frac{3e^{-3s}}{s^2}-\frac{2e^{-4s}}{s^2}$.

Section 4.3

1. $e^{t-3}h(t-3)=\begin{cases}0 & \text{if } 0\le t<3,\\ e^{t-3} & \text{if } t\ge 3.\end{cases}$

2. $(t-3)h(t-3)=\begin{cases}0 & \text{if } 0\le t<3,\\ t-3 & \text{if } t\ge 3.\end{cases}$

3. $\frac{1}{2}(t-3)^2e^{t-3}h(t-3)=\begin{cases}0 & \text{if } 0\le t<3,\\ \frac{1}{2}(t-3)^2e^{t-3} & \text{if } t\ge 3.\end{cases}$

4. $h(t-\pi)\sin(t-\pi)=\begin{cases}0 & \text{if } 0\le t<\pi,\\ \sin(t-\pi) & \text{if } t\ge \pi\end{cases}=\begin{cases}0 & \text{if } 0\le t<\pi,\\ -\cos t & \text{if } t\ge \pi.\end{cases}$

5. $h(t-3\pi)\cos(t-3\pi)=\begin{cases}0 & \text{if } 0\le t<3\pi,\\ \cos(t-3\pi) & \text{if } t\ge 3\pi\end{cases}=\begin{cases}0 & \text{if } 0\le t<3\pi,\\ -\cos t & \text{if } t\ge 3\pi.\end{cases}$

6. $\frac{1}{2}e^{-(t-\pi)}\sin 2(t-\pi)h(t-\pi)=\begin{cases}0 & \text{if } 0\le t<\pi,\\ \frac{1}{2}e^{-(t-\pi)}\sin 2t & \text{if } t\ge \pi.\end{cases}$

7. $(t-1)h(t-1)+\frac{1}{2}(t-2)^2e^{t-2}h(t-2)$

8. $\frac{1}{2}h(t-2)\sin 2(t-2)=\begin{cases}0 & \text{if } 0\le t<2,\\ \frac{1}{2}\sin 2(t-2) & \text{if } t\ge 2.\end{cases}$

9. $\frac{1}{4}h(t-2)\left(e^{2(t-2)}-e^{-2(t-2)}\right)=\begin{cases}0 & \text{if } 0\le t<2,\\ \frac{1}{4}\left(e^{2(t-2)}-e^{-2(t-2)}\right) & \text{if } t\ge 2.\end{cases}$

10. $h(t-5)\left(2e^{-2(t-5)}-e^{-(t-5)}\right)=\begin{cases}0 & \text{if } 0\le t<5,\\ 2e^{-2(t-5)}-e^{-(t-5)} & \text{if } t\ge 5.\end{cases}$

11. $h(t-2)\left(e^{2(t-2)}-e^{t-2}\right)+h(t-3)\left(e^{2(t-3)}-e^{t-3}\right)$

12. $t-(t-5)h(t-5)=\begin{cases}t & \text{if } 0\le t<5,\\ 5 & \text{if } t\ge 5.\end{cases}$

13. $\frac{1}{6}t^3+\frac{1}{6}(t-3)^3h(t-3)=\begin{cases}\frac{1}{6}t^3 & \text{if } 0\le t<3,\\ \frac{1}{6}t^3+\frac{1}{6}(t-3)^3 & \text{if } t\ge 3.\end{cases}$

14. $h(t-\pi)e^{-3(t-\pi)}\left(2\cos 2(t-\pi) - \frac{5}{2}\sin 2(t-\pi)\right) = \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ e^{-3(t-\pi)}\left(2\cos 2t - \frac{5}{2}\sin 2t\right) & \text{if } t \ge \pi. \end{cases}$

15. $e^{-3t}\left(2\cos 2t - \frac{5}{2}\sin 2t\right) - h(t-\pi)e^{-3(t-\pi)}\left(2\cos 2(t-\pi) - \frac{5}{2}\sin 2(t-\pi)\right)$

Section 4.4

1. $y = -\frac{3}{2}h(t-1)\left(1 - e^{-2(t-1)}\right)$
2. $y = e^{-2t} - 1 + 2h(t-1)\left(1 - e^{-2(t-1)}\right)$
3. $y = h(t-1)\left(1 - e^{-2(t-1)}\right) - h(t-3)\left(1 - e^{-2(t-3)}\right)$
4. $y = \frac{1}{4}e^{-2t} - \frac{1}{4} + \frac{1}{2}t + h(t-1)\left(\frac{1}{4}e^{-2(t-1)} - \frac{1}{4} + \frac{1}{2}(t-1)\right) - \frac{1}{2}h(t-1)\left(1 - e^{-2(t-1)}\right)$
5. $-\frac{1}{9}h(t-3)\left(-1 + \cos 3(t-3)\right)$
6. $y = -\frac{2}{3}e^{t} + \frac{5}{12}e^{4t} + \frac{1}{4} + \frac{1}{12}h(t-5)\left(-3 + 4e^{t-5} - e^{4(t-5)}\right)$
7. $y = \frac{1}{3}h(t-1)\left(1 - 3e^{-2(t-1)} + 2e^{-3(t-1)}\right) + \frac{1}{3}h(t-3)\left(-1 + 3e^{-3(t-3)} - 2e^{-3(t-3)}\right)$
8. $y = \cos 3t + \frac{1}{24}h(t-2\pi)\left(3\sin t - \sin 3t\right)$
9. $y = te^{-t} + h(t-3)\left(1 - (t-2)e^{-(t-3)}\right)$
10. $y = te^{-t} - \frac{1}{4}h(t-3)\left(-e^{t} - 5e^{-t+6} + 2te^{-t+6}\right)$
11. $y = \frac{1}{20}e^{-5t} - \frac{1}{4}e^{-t} + \frac{1}{5} + \frac{1}{20}h(t-2)\left(4 + e^{-5(t-2)} - 5e^{-(t-2)}\right) + \frac{1}{20}h(t-4)\left(4 + e^{-5(t-4)} - 5e^{-(t-4)}\right) + \frac{1}{20}h(t-6)\left(4 + e^{-5(t-6)} - 5e^{-(t-6)}\right)$

Sections 4.5 and 4.6

1. $y = h(t-1)e^{-2(t-1)}$
2. $y = (1 + h(t-1))e^{-2(t-1)}$
3. $y = h(t-1)e^{-2(t-1)} - h(t-3)e^{-2(t-3)}$
4. $y = \frac{1}{2}\left(1 + h(t-\pi)\right)\sin 2t = \begin{cases} \frac{1}{2}\sin 2t & \text{if } 0 \le t < \pi, \\ \sin 2t & \text{if } t \ge \pi. \end{cases}$
5. $y = \frac{1}{2}\chi_{[\pi,\, 2\pi)}\sin 2t = \begin{cases} \frac{1}{2}\sin 2t & \text{if } \pi \le t < 2\pi, \\ 0 & \text{otherwise.} \end{cases}$
6. $y = \cos 2t + \frac{1}{2}\chi_{[\pi,\, 2\pi)}\sin 2t$

7. $y = (t-1)e^{-2(t-1)}h(t-1)$
8. $y = (t-1)\left(e^{-2t} + e^{-2(t-1)}h(t-1)\right)$
9. $y = 3h(t-1)e^{-2(t-1)}\sin(t-1)$
10. $y = e^{-2t}(\sin t - \cos t) + 3h(t-1)e^{-2(t-1)}\sin(t-1)$
11. $y = e^{-2t}\left(\cos 4t + \frac{1}{2}\sin 4t\right) + \frac{1}{4}\sin 4t\left(h(t-\pi)e^{-2(t-\pi)} - h(t-2\pi)e^{-2(t-2\pi)}\right)$
12. $y = \frac{1}{18}\left(e^{5t} - e^{-t} - 6te^{-t}\right) + \frac{1}{6}h(t-3)\left(e^{5(t-3)} - e^{-(t-3)}\right)$

Chapter 5

Section 5.1

2. $AB = \begin{bmatrix} -3 & 1 \\ -3 & 5 \end{bmatrix}$, $AC = \begin{bmatrix} 6 & 3 \\ 4 & 6 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & -1 & -1 \\ 5 & -2 & 18 \\ 0 & 1 & 5 \end{bmatrix}$, $CA = \begin{bmatrix} 2 & 0 & 8 \\ -2 & 3 & 7 \\ 3 & -1 & 7 \end{bmatrix}$
3. $A(B+C) = AB + AC = \begin{bmatrix} 3 & 4 \\ 1 & 11 \end{bmatrix}$, $(B+C)A = \begin{bmatrix} 3 & -1 & 7 \\ 3 & 1 & 25 \\ 3 & 2 & 12 \end{bmatrix}$
4. $C = \begin{bmatrix} -2 & 5 \\ -13 & -8 \\ 7 & 0 \end{bmatrix}$
5. $AB = \begin{bmatrix} 6 & 4 & -1 & -8 \\ 0 & 2 & -8 & 2 \\ 2 & -1 & 9 & -5 \end{bmatrix}$
6. $BC = \begin{bmatrix} 2 & 3 & -8 \\ -2 & 0 & 24 \end{bmatrix}$
7. $CA = \begin{bmatrix} 8 & 0 \\ 4 & -5 \\ 8 & 14 \\ 10 & 11 \end{bmatrix}$
8. $B^tA^t = \begin{bmatrix} 6 & 0 & 2 \\ 4 & 2 & -1 \\ -1 & -8 & 9 \\ -8 & 2 & -5 \end{bmatrix}$

9. $ABC = \begin{bmatrix} 8 & 9 & -48 \\ 4 & 0 & -48 \\ -2 & 3 & 40 \end{bmatrix}$.

10. $AB = -4$ and $BA = \begin{bmatrix} 1 & 4 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & -3 & -1 \\ -2 & -8 & -6 & -2 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

15. $\begin{bmatrix} 0 & 0 & -1 \\ 3 & -5 & -1 \\ 0 & 0 & 5 \end{bmatrix}$

16. $AB - BA = \begin{bmatrix} ab & 0 \\ 0 & -ab \end{bmatrix}$. It is not possible to have $ab = 1$ and $-ab = 1$ since $1 \neq -1$.

17. (a) Choose, for example, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(b) $(A+B)^2 = A^2 + 2AB + B^2$ precisely when $AB = BA$.

18. $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\quad A^3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

19. $B^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

20. $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$

21. (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$; the two rows of A are switched. (b) $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} v_1 + cv_2 \\ v_2 \end{bmatrix}$; to the first row is added c times the second row while the second row is unchanged, (c) to the second row is added c times the first row while the first row is unchanged. (d) the first row is multiplied by a while the second row is unchanged, (e) the second row is multiplied by a while the first row is unchanged.

Section 5.2

1. (a) $A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 6 \end{bmatrix}$, and $[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 1 & 1 & -1 & 4 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 6 \end{array}\right]$.

   (b) $A = \begin{bmatrix} 2 & -3 & 4 & 1 \\ 3 & 8 & -3 & -6 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $[A|\mathbf{b}] = \left[\begin{array}{cccc|c} 2 & -3 & 4 & 1 & 0 \\ 3 & 8 & -3 & -6 & 1 \end{array}\right]$.

2. $$\begin{array}{rcrcrcrcrcr} x_1 & - & & & x_3 & + & 4x_4 & + & 3x_5 & = & 2 \\ 5x_1 & + & 3x_2 & - & 3x_3 & - & x_4 & - & 3x_5 & = & 1 \\ 3x_1 & - & 2x_2 & + & 8x_3 & + & 4x_4 & - & 3x_5 & = & 3 \\ -8x_1 & + & 2x_2 & & & + & 2x_4 & + & x_5 & = & -4 \end{array}$$

3. $p_{2,3}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

4. RREF

5. $t_{2,1}(-2)(A) = \begin{bmatrix} 1 & 0 & -5 & -2 & -1 \\ 0 & 1 & 3 & 1 & 1 \end{bmatrix}$

6. $m_2(1/2)(A) = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. RREF

8. $t_{1,3}(-3)(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 0 & -11 & -8 \\ 0 & 1 & 0 & -4 & -2 \\ 0 & 0 & 1 & 9 & 6 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 1 & 0 & \frac{7}{2} & \frac{1}{4} \\ 0 & 0 & 1 & 3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & `1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

18. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}$

19. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

20. $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

21. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 5 \end{bmatrix} + \alpha \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

22. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

23. no solution

24. $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

25. The equation $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ has solution $a = 2$ and $b = 3$. By Proposition 5.2.6 $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ is a solution.

26. $k = 2$

27. (a) If $\mathbf{x}_i$ is the solution set for $A\mathbf{x} = \mathbf{b}_i$ then $\mathbf{x}_1 = \begin{bmatrix} -7/2 \\ 7/2 \\ -3/2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -3/2 \\ 3/2 \\ -1/2 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 7 \\ -6 \\ 3 \end{bmatrix}$.

(b) The augmented matrix $[A|b_1|b_2|b_3]$ reduces to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/2 & -3/2 & 7 \\ 0 & 1 & 0 & 7/2 & 3/2 & -6 \\ 0 & 0 & 1 & -3/2 & -1/2 & 3 \end{array}\right].$$

The last three columns correspond in order to the solutions.

Section 5.3

1. $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$

3. not invertible

4. $\begin{bmatrix} -2 & 1 \\ -3/2 & 1/2 \end{bmatrix}$

5. not invertible

6. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} -6 & 5 & 13 \\ 5 & -4 & -11 \\ -1 & 1 & 3 \end{bmatrix}$

8. $\begin{bmatrix} -1/5 & 2/5 & 2/5 \\ -1/5 & -1/10 & 2/5 \\ -3/5 & 1/5 & 1/5 \end{bmatrix}$

9. $\begin{bmatrix} -29 & 39/2 & -22 & 13 \\ 7 & -9/2 & 5 & -3 \\ -22 & 29/2 & -17 & 10 \\ 9 & -6 & 7 & -4 \end{bmatrix}$

10. $\frac{1}{2}\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 \end{bmatrix}$

12. not invertible

13. $\mathbf{b} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

14. $\mathbf{b} = \begin{bmatrix} -2 \\ 6 \\ -3 \end{bmatrix}$

15. $\mathbf{b} = \frac{1}{10} \begin{bmatrix} 16 \\ 11 \\ 18 \end{bmatrix}$

16. $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

17. $\mathbf{b} = \begin{bmatrix} 19 \\ -4 \\ 15 \\ -6 \end{bmatrix}$

18. $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 1 \end{bmatrix}.$

19. $(A^t)^{-1} = (A^{-1})^t$

20. $(E(\theta))^{-1} = E(-\theta)$

21. $F(\theta)^{-1} = F(-\theta)$

22.

Section 5.4

1. 1
2. 0
3. 10
4. 8
5. -21
6. 6

7. 2

8. 15

9. 0

10. $\frac{1}{s^2-3s}\begin{bmatrix} -2+s & 2 \\ 1 & -1+s \end{bmatrix}$ $s = 0, 3$

11. $\frac{1}{s^2-6s+8}\begin{bmatrix} s-3 & 1 \\ 1 & s-3 \end{bmatrix}$ $s = 2, 4$

12. $\frac{1}{s^2-2s+s}\begin{bmatrix} s-1 & 1 \\ -1 & s-1 \end{bmatrix}$ $s = 1 \pm i$

13. $\frac{1}{(s-1)^3}\begin{bmatrix} (s-1)^2 & 3 & s-1 \\ 0 & (s-1)^2 & 0 \\ 0 & 3(s-1) & (s-1)^2 \end{bmatrix}$ $s = 1$

14. $\frac{1}{s^3-3s^2-6s+8}\begin{bmatrix} s^2-2s+10 & -3s-6 & 3s-12 \\ -3s+12 & s^2-2s-8 & 3s-12 \\ 3s+6 & -3s-6 & s^2-2s-8 \end{bmatrix}$ $s = -2, 1, 4$

15. $\frac{1}{s^3+s^2+4s+4}\begin{bmatrix} s^2+s & 4s+4 & 0 \\ -s-1 & s^2+s & 0 \\ s-4 & 4s+4 & s^2+4 \end{bmatrix}$ $s = -1, \pm 2i$

16. $\begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$

17. no inverse

18. $\frac{1}{10}\begin{bmatrix} 6 & -4 \\ -2 & 3 \end{bmatrix}$

19. $\frac{1}{8}\begin{bmatrix} 4 & -4 & 4 \\ -1 & 3 & -1 \\ -5 & -1 & 3 \end{bmatrix}$

20. $\frac{1}{21}\begin{bmatrix} 27 & -12 & 3 \\ -13 & 5 & 4 \\ -29 & 16 & -4 \end{bmatrix}$

21. $\frac{1}{6}\begin{bmatrix} 2 & -98 & 9502 \\ 0 & 3 & -297 \\ 0 & 0 & 6 \end{bmatrix}$

22. $\frac{1}{2}\begin{bmatrix} -13 & 76 & -80 & 35 \\ -14 & 76 & -80 & 36 \\ 6 & -34 & 36 & -16 \\ 7 & -36 & 38 & -17 \end{bmatrix}$

23. $\frac{1}{15}\begin{bmatrix} 55 & -95 & 44 & -171 \\ 50 & -85 & 40 & -150 \\ 70 & -125 & 59 & -216 \\ 65 & -115 & 52 & -198 \end{bmatrix}$

24. no inverse

Chapter 6

Section 6.1

1. nonlinear, autonomous
2. linear, constant coefficient, not autonomous, not homogeneous
3. linear, homogeneous, but not constant coefficient or autonomous
4. nonlinear and not autonomous
5. linear, constant coefficient, homogeneous, and autonomous
6. linear, constant coefficient, not homogeneous, but autonomous

In all of the following solutions, $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$.

12. $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

13. $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ k^2 & 0 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

14. $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 0 \\ A\cos\omega t \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

15. $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

16. $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 0 \\ A\sin\omega t \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

17. $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{t^2} & -\dfrac{2}{t} \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(1) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Section 6.2

1. $A'(t) = \begin{bmatrix} -2\sin 2t & 2\cos 2t \\ -2\cos 2t & -2\sin 2t \end{bmatrix}$

2. $A'(t) = \begin{bmatrix} -3e^{-3t} & 1 \\ 2t & 2e^{2t} \end{bmatrix}$

3. $A'(t) = \begin{bmatrix} -e^{-t} & (1-t)e^{-t} & (2t-t^2)e^{-t} \\ 0 & -e^{-t} & (1-t)e^{-t} \\ 0 & 0 & -e^{-t} \end{bmatrix}$

4. $\boldsymbol{y}'(t) = \begin{bmatrix} 1 \\ 2t \\ t^{-1} \end{bmatrix}$

5. $A'(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

6. $\boldsymbol{v}'(t) = \begin{bmatrix} -2e^{-2t} & \frac{2t}{t^2+1} & -3\sin 3t \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

8. $\frac{1}{4}\begin{bmatrix} e^2 - e^{-2} & e^2 + e^{-2} - 1 \\ 1 - e^2 - e^{-2} & e^2 - e^{-2} \end{bmatrix}$

9. $\begin{bmatrix} 3/2 \\ 7/3 \\ \ln 4 - 1 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix}$

11. Continuous on I_1, I_4, and I_5.

12. $\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ \frac{2}{s^3} & \frac{1}{s-2} \end{bmatrix}$

13. $\begin{bmatrix} \frac{2}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$

14. $\begin{bmatrix} \frac{3!}{s^4} & \frac{2s}{(s^2+1)^2} & \frac{1}{(s+1)^2} \\ \frac{2-s}{s^3} & \frac{s-3}{s^2-6s+13} & \frac{3}{s} \end{bmatrix}$

15. $\begin{bmatrix} \frac{1}{s^2} \\ \frac{2}{s^3} \\ \frac{6}{s^4} \end{bmatrix}$

16. $\frac{2}{s^2-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

17. $\begin{bmatrix} \frac{1}{s} & \frac{1}{s2+1} & \frac{1}{s(s^2+1)} \\ 0 & \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ 0 & \frac{-1}{s+1} & \frac{s}{s^2+1} \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2t & 3t^2 \end{bmatrix}$

19. $\begin{bmatrix} 1 & t \\ \frac{e^t+e^{-t}}{2} & \cos t \end{bmatrix}$

20. $\begin{bmatrix} e^t & te^t \\ \frac{-4}{3} + \frac{e^{-3t}}{3} + e^t & \sin t \\ 3\cos 3t & e^{3t} \end{bmatrix}$

21. $\begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}$

22. $(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{bmatrix}$ and $\mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} e^t & 0 \\ 0 & e^2t \end{bmatrix}$

23. $(sI-A)^{-1} = \begin{bmatrix} \frac{s-2}{s(s-3)} & \frac{-1}{s(s-3)} \\ \frac{-2}{s(s-3)} & \frac{s-1}{s(s-3)} \end{bmatrix}$ and $\mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} \frac{2}{3}+\frac{1}{3}e^{3t} & \frac{1}{3}-\frac{1}{3}e^{3t} \\ \frac{2}{3}-\frac{2}{3}e^{3t} & \frac{1}{3}+\frac{2}{3}e^{3t} \end{bmatrix}$

24. $(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{s+1}{s^3} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$ and $\mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} 1 & t & t+\frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$

25. $(sI-A)^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$ and $\mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

26. (a) $\mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1+(\frac{t^2}{2}) \\ 1+(\frac{t^2}{2}) \end{bmatrix}$, $\mathbf{y}_2 = \begin{bmatrix} 1+(\frac{t^2}{2})+\frac{1}{2}(\frac{t^2}{2})^2 \\ 1+(\frac{t^2}{2})+\frac{1}{2}(\frac{t^2}{2})^2 \end{bmatrix}$,
$\mathbf{y}_3 = \begin{bmatrix} 1+(\frac{t^2}{2})+\frac{1}{2}(\frac{t^2}{2})^2+\frac{1}{3!}(\frac{t^2}{2})^3 \\ 1+(\frac{t^2}{2})+\frac{1}{2}(\frac{t^2}{2})^2+\frac{1}{3!}(\frac{t^2}{2})^3 \end{bmatrix}$.

(b) The n^{th} term is

$$\mathbf{y} = \begin{bmatrix} 1+(\frac{t^2}{2})+\cdots+\frac{1}{n!}(\frac{t^2}{2})^n \\ 1+(\frac{t^2}{2})+\cdots+\frac{1}{n!}(\frac{t^2}{2})^n \end{bmatrix}$$

(c) By the Existence and Uniqueness theorem there are no other solutions.

27. (a) $\mathbf{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ -(\frac{t^2}{2}) \end{bmatrix}$, $\mathbf{y}_2 = \begin{bmatrix} 1-\frac{1}{2}(\frac{t^2}{2})^2 \\ -(\frac{t^2}{2}) \end{bmatrix}$,
$\mathbf{y}_3 = \begin{bmatrix} 1-\frac{1}{2}(\frac{t^2}{2})^2 \\ -(\frac{t^2}{2})+\frac{1}{3!}(\frac{t^2}{2})^3 \end{bmatrix}$.

(b) By the Uniqueness and Existence Theorem there are no other solutions.

28. (a) $\mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1+t^2 \\ 1-t^2 \end{bmatrix}$,
$\mathbf{y}_2 = \begin{bmatrix} 1+t^2 \\ 1-t^2 \end{bmatrix}$, $\mathbf{y}_3 = \begin{bmatrix} 1+t^2 \\ 1-t^2 \end{bmatrix}$.

(b) $\mathbf{y} = \begin{bmatrix} 1+t^2 \\ 1-t^2 \end{bmatrix}$.

30. $(-\infty, \infty)$

31. $(-2, 3)$

32. $(-\infty, 2)$

33. $(-\infty, \infty)$

34. (b) $e^{Nt} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$. (c) $\boldsymbol{y}(t) = \begin{bmatrix} 1+2t+\frac{3t^2}{2} \\ 2+3t \\ 3 \end{bmatrix}$.

(d) and (e): Both matrices are $\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{1}{s^3} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$

35. (c) (iv) $\boldsymbol{y}(t) = \begin{bmatrix} (c_1 + c_2 t)e^t \\ c_2 e^t \end{bmatrix}$.

(d) and (e): Both matrices are $\begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}$

Section 6.3

1. All except (b) and (e) are fundamental matrices.

2.

A	$\boldsymbol{\Psi}(t) = e^{At}$	$\boldsymbol{y}(t)$
(a)	$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$	$\begin{bmatrix} 3\cos t - 2\sin t \\ -3\sin t + 2\cos t \end{bmatrix}$
(c)	$\frac{1}{3}\begin{bmatrix} 4e^{-t} - e^{2t} & -e^{-t} + e^{2t} \\ 4e^{-t} - 4e^{2t} & -e^{-t} + 4e^{2t} \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} 14e^{-t} - 5e^{2t} \\ 14e^{-t} - 20e^{2t} \end{bmatrix}$
(f)	$\begin{bmatrix} -2e^{2t} + 3e^{3t} & 3e^{2t} - 3e^{3t} \\ -2e^{2t} + 2e^{3t} & 3e^{2t} - 2e^{3t} \end{bmatrix}$	$\begin{bmatrix} -12e^{2t} + 15e^{3t} \\ -12e^{2t} + 10e^{3t} \end{bmatrix}$
(g)	$\frac{1}{4}\begin{bmatrix} 3e^{2t} + e^{6t} & -3e^{2t} + 3e^{6t} \\ -e^{2t} + e^{6t} & e^{2t} + 3e^{6t} \end{bmatrix}$	$\frac{1}{4}\begin{bmatrix} 15e^{2t} - 3e^{6t} \\ -5e^{2t} - 3e^{6t} \end{bmatrix}$
(h)	$\begin{bmatrix} (1-2t)e^{3t} & -4te^{3t} \\ te^{3t} & (1+2t)e^{3t} \end{bmatrix}$	$\begin{bmatrix} (3+2t)e^{3t} \\ -(2+t)e^{3t} \end{bmatrix}$

3. $A(t)$ $\quad\quad$ $\boldsymbol{\Psi}(t)$ $\quad\quad$ $\boldsymbol{y}(t)$

(i) $\begin{bmatrix} \cos(t^2/2) & -\sin(t^2/2) \\ \sin(t^2/2) & \cos(t^2/2) \end{bmatrix}$ $\quad$ $\begin{bmatrix} 3\cos(t^2/2) - 2\sin(t^2/2) \\ -3\sin(t^2/2) - 2\cos(t^2/2) \end{bmatrix}$

(j) $\frac{1}{4}\begin{bmatrix} 4+2t^2 & 2t^2 \\ -2t^2 & 4-2t^2 \end{bmatrix}$ $\quad$ $\frac{1}{4}\begin{bmatrix} 12+2t^2 \\ -8-2t^2 \end{bmatrix}$

(k) $\begin{bmatrix} \frac{1}{2}(e^{t^2/2}+e^{-t^2/2}) & \frac{1}{2}(e^{t^2/2}-e^{-t^2/2}) \\ \frac{1}{2}(e^{t^2/2}-e^{-t^2/2}) & \frac{1}{2}(e^{t^2/2}+e^{-t^2/2}) \end{bmatrix}$ $\quad$ $\begin{bmatrix} \frac{1}{2}e^{t^2/2}+\frac{5}{2}e^{-t^2/2} \\ \frac{1}{2}e^{t^2/2}-\frac{5}{2}e^{-t^2/2} \end{bmatrix}$

4. (a), (c), and (d) are linearly independent.

Section 6.4

In the following, c_1, c_2, and c_3 denote arbitrary real constants.

1. (a) $\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix}$; (b) $\boldsymbol{y}(t) = \begin{bmatrix} c_1e^{-t} \\ c_2e^{3t} \end{bmatrix}$

2. (a) $\begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$; (b) $\begin{bmatrix} c_1\cos 2t + c_2\sin 2t \\ -c_1\sin 2t + c_2\cos 2t \end{bmatrix}$

3. (a) $\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$; (b) $\begin{bmatrix} c_1e^{2t} + c_2e^{2t} \\ c_2e^{2t} \end{bmatrix}$

4. (a) $\begin{bmatrix} e^{-t}\cos 2t & e^{-t}\sin 2t \\ -e^{-t}\sin 2t & e^{-t}\cos 2t \end{bmatrix}$; (b) $\begin{bmatrix} c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t \\ -c_1e^{-t}\sin 2t + c_2e^{-t}\cos 2t \end{bmatrix}$

5. (a) $\frac{1}{2}\begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix}$; (b) $\frac{1}{2}\begin{bmatrix} (3c_1 - c_2)e^t + (-c_2 + c_2)e^{-t} \\ (3c_1 - c_2)e^t + 3(-c_1 + c_2)e^{-t} \end{bmatrix}$

6. (a) $\begin{bmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t - 2te^t \end{bmatrix}$; (b) $\begin{bmatrix} c_1e^t + (2c_1 - 4c_2)te^t \\ c_2e^t + (c_1 - 2c_2)te^t \end{bmatrix}$

7. (a) $\begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{bmatrix}$; (b) $\begin{bmatrix} c_2\cos t + (2c_1 - 5c_2)\sin t \\ c_2\cos t + (c_1 - 2c_2)\sin t \end{bmatrix}$

8. (a) $\begin{bmatrix} e^{-t}\cos 2t & -2e^{-t}\sin 2t \\ \frac{1}{2}e^{-t}\sin 2t & e^{-t}\cos 2t \end{bmatrix}$; (b) $\begin{bmatrix} e^{-t}(c_1\cos 2t - 2c_2\sin 2t) \\ \frac{1}{2}e^{-t}(c_1\sin 2t + 2\cos 2t) \end{bmatrix}$

9. (a) $\frac{1}{2}\begin{bmatrix} e^t + e^{3t} & e^{3t} - e^t \\ e^{3t} - e^t & e^t + e^{3t} \end{bmatrix}$; (b) $\frac{1}{2}\begin{bmatrix} (c_1 - c_2)e^t + (c_1 + c_2)e^{3t} \\ (c_1 + c_2)e^{3t} + (c_2 - c_1)e^t \end{bmatrix}$

10. (a) $\begin{bmatrix} e^t+4te^t & 2te^t \\ -8te^t & e^t-4te^t \end{bmatrix}$; (b) $\begin{bmatrix} c_1e^t+(4c_1+2c_2)te^t \\ c_2e^t-(8c_1+4c_2)te^t \end{bmatrix}$

11. (a) $\begin{bmatrix} e^{-t} & 0 & \frac{3}{2}(e^t-e^{-t}) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{bmatrix}$; (b) $\begin{bmatrix} (c_1-\frac{3}{2}c_3)e^{-t}+\frac{3}{2}c_3e^t \\ c_2e^{2t} \\ c_3e^t \end{bmatrix}$

12. (a) $\begin{bmatrix} \cos 2t & 2\sin 2t & 0 \\ -\frac{1}{2}\sin 2t & \cos 2t & 0 \\ -e^{-t}+\cos 2t & 2\sin 2t & e^{-t} \end{bmatrix}$; (b) $\begin{bmatrix} c_1\cos 2t+2c_2\sin 2t \\ -\frac{1}{2}c_1\sin 2t+c_2\cos 2t \\ (c_3-c_1)e^{-t}+c_1\cos 2t+2c_2\sin 2t \end{bmatrix}$

13. (a) $\frac{1}{3}\begin{bmatrix} 2e^{-3t}+1 & 3e^{-t}-3e^{-3t} & 1-e^{-3t} \\ 0 & 3e^{-t} & 0 \\ 2-2e^{-3t} & -3e^{-t}+3e^{-3t} & e^{-3t}+2 \end{bmatrix}$;
(b) $\frac{1}{3}\begin{bmatrix} (2c_1-3c_2-c_3)e^{-3t}+3c_2e^{-t}+(c_1+c_3) \\ c_2e^{-t} \\ (-2c_1+3c_2+c_3)e^{-3t}-3c_2e^{-t}+(2c_1+2c_3) \end{bmatrix}$

14. (a) $\begin{bmatrix} -te^t+e^t & te^t & te^t \\ e^{2t}-e^t & e^t & -e^{2t}+e^t \\ -te^t-e^{2t}+e^t & te^t & e^{2t}+te^t \end{bmatrix}$; (b) $\begin{bmatrix} c_1e^t+(c_2+c_3-c_1)te^t \\ (-c_1+c_2+c_3)e^t+(c_1-c_3)e^{2t} \\ c_1e^t+(-c_1+c_2+c_3)te^t+(c_3-c_1)e^{2t} \end{bmatrix}$

15. (a) $\begin{bmatrix} e^{3t} & te^{3t} & -\frac{1}{2}t^2e^{3t}-te^{3t} \\ 0 & e^{3t} & -te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}$; (b) $\begin{bmatrix} c_1e^{3t}+(c_1-c_3)e^{3t}-\frac{1}{2}c_3t^2e^{3t} \\ c_2e^{3t}-c_3te^{3t} \\ c_3e^{3t} \end{bmatrix}$

Section 6.6

1. $\begin{bmatrix} te^{-t} \\ e^{3t}-e^t \end{bmatrix}$

2. $\frac{1}{3}\begin{bmatrix} 2\cos t-2\cos 2t \\ 2\sin 2t-\sin t \end{bmatrix}$

3. $\frac{1}{2}\begin{bmatrix} t(e^{2t}-1) \\ e^{2t}-1 \end{bmatrix}$

5. $\frac{1}{2}\begin{bmatrix} 3te^t+te^{-t}+e^{-t}-e^t \\ 3te^t+3te^{-t}+2e^{-t}-2e^t \end{bmatrix}$

7. $\frac{1}{2}\begin{bmatrix} 5t\cos t-5\sin t \\ t\sin t+2t\cos t-2\sin t \end{bmatrix}$

11. $\frac{1}{4}\begin{bmatrix} 5e^t-5e^{-t}-6te^{-t} \\ 4te^{2t} \\ 2e^t-2e^{-t} \end{bmatrix}$

Appendix A

Section A.1

1. $z = (1,1) = 1 + i = [\frac{\pi}{4}, \sqrt{2}] = \sqrt{2}e^{i\frac{\pi}{4}}$, $w = (-1,1) = -1 + i = [\frac{3\pi}{4}, \sqrt{2}] = \sqrt{2}e^{i\frac{3\pi}{4}}$, $z \cdot w = -2$, $\frac{z}{w} = -i$, $\frac{w}{z} = i$, $z^2 = 2i$, $\sqrt{z} = \pm(\sqrt{\sqrt{2}}\cos(\frac{\pi}{8}) + i\sqrt{\sqrt{2}}\sin(\frac{\pi}{8})) = \pm(\frac{1}{2}\sqrt{2\sqrt{2}+2} + i\frac{1}{2}\sqrt{2\sqrt{2}-2})$ since $\cos(\frac{\pi}{8}) = \frac{1}{2}\sqrt{2+\sqrt{2}}$ and $\sin(\frac{\pi}{8}) = \frac{1}{2}\sqrt{2-\sqrt{2}}$, $z^{11} = -32 + 32i$.

2. (a) $-5 + 10i$ (b) $-3 + 4i$ (c) $\frac{2}{13} - \frac{3}{13}i$ (d) $\frac{8}{130} - \frac{1}{130}i$ (e) $\frac{6}{5} - \frac{8}{5}i$.

3. (a) - (c) check your result (d) $z = (3\pi/2 + 2k\pi)i$ for all integers k.

4. Always either 5 or 1.

5. The vertical line $x = \dfrac{3}{2}$. The distance between two points z, w in the plane is given by $|z - w|$. Hence, the equation describes the set of points z in the plane which are equidistant from 1 and 2.

6. This is the set of points inside the ellipse with foci $(1,0)$ and $(3,0)$ and major axis of length 4.

7. (a) $\pm\left(\sqrt{\sqrt{2}+1} + i\sqrt{\sqrt{2}-1}\right)$ (b) $\pm(2+i)$

8. (a) $5e^{i\tan^{-1}(4/3)} \approx 5e^{0.927i}$ (b) $5e^{-i\tan^{-1}(4/3)}$ (c) $25e^{2i\tan^{-1}(4/3)}$ (d) $\frac{1}{5}e^{-i\tan^{-1}(4/3)}$ (e) $5e^{i\pi}$ (f) $3e^{\frac{i\pi}{2}}$

9. (a) Real: $2e^{-t}\cos t - 3e^{-t}\sin t$; Imaginary: $3e^{-t}\cos t + 2e^{-t}\sin t$ (b) Real: $-e^{\pi}\sin 2t$; Imaginary: $e^{\pi}\cos 2t$ (c) Real: $e^{-t}\cos 2t$; Imaginary: $e^{-t}\sin 2t$

10. (a) If $z = 2\pi ki$ for k an integer, the sum is n. Otherwise the sum is $\dfrac{1-e^{nz}}{1-e^{z}}$. (b) 0

11. $-2i$, $\sqrt{3} + i$, $-\sqrt{3} + 2i$

Appendix C

Tables

Table C.1: **Laplace Transform Rules**

	$f(t)$	$F(s)$
1.	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
2.	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
3.	$e^{at} f(t)$	$F(s-a)$
4.	$f(t-c)h(t-c)$	$e^{-sc}F(s)$
5.	$f'(t)$	$sF(s) - f(0)$
6.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
7.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
8.	$tf(t)$	$-F'(s)$
9.	$t^2 f(t)$	$F''(s)$
10.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
11.	$\frac{1}{t} f(t)$	$\int_s^\infty F(u)du$
12.	$\int_0^t f(v)dv$	$\frac{F(s)}{s}$
13.	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$

Table C.2: **Table of Laplace Transforms**

	$F(s)$	$f(t)$
1.	$\dfrac{1}{s}$	1
2.	$\dfrac{1}{s^2}$	t
3.	$\dfrac{1}{s^n} \quad (n = 1, 2, 3, \ldots)$	$\dfrac{t^{n-1}}{(n-1)!}$
4.	$\dfrac{1}{s-a}$	e^{at}
5.	$\dfrac{1}{(s-a)^2}$	te^{at}
6.	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, 3, \ldots)$	$\dfrac{t^{n-1}e^{at}}{(n-1)!}$
7.	$\dfrac{b}{s^2+b^2}$	$\sin bt$
8.	$\dfrac{s}{s^2+b^2}$	$\cos bt$
9.	$\dfrac{b}{(s-a)^2+b^2}$	$e^{at}\sin bt$
10.	$\dfrac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$
11.	$\operatorname{Re}\left(\dfrac{n!}{(s-(a+bi))^{n+1}}\right) \quad (n = 0, 1, 2, \ldots)$	$t^n e^{at}\cos bt$
12.	$\operatorname{Im}\left(\dfrac{n!}{(s-(a+bi))^{n+1}}\right) \quad (n = 0, 1, 2, \ldots)$	$t^n e^{at}\sin bt$
13.	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{e^{at}-e^{bt}}{a-b}$
14.	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{ae^{at}-be^{bt}}{a-b}$

Table C.2: **Table of Laplace Transforms**

	$F(s)$	$f(t)$
15.	$\dfrac{1}{(s-a)(s-b)(s-c)}$ a, b, c distinct	$\dfrac{e^{at}}{(a-b)(a-c)} + \dfrac{e^{bt}}{(b-a)(b-c)} + \dfrac{e^{ct}}{(c-a)(c-b)}$
16.	$\dfrac{s}{(s-a)(s-b)(s-c)}$ a, b, c distinct	$\dfrac{ae^{at}}{(a-b)(a-c)} + \dfrac{be^{bt}}{(b-a)(b-c)} + \dfrac{ce^{ct}}{(c-a)(c-b)}$
17.	$\dfrac{s^2}{(s-a)(s-b)(s-c)}$ a, b, c distinct	$\dfrac{a^2e^{at}}{(a-b)(a-c)} + \dfrac{b^2e^{bt}}{(b-a)(b-c)} + \dfrac{c^2e^{ct}}{(c-a)(c-b)}$
18.	$\dfrac{s}{(s-a)^2}$	$(1+at)e^{at}$
19.	$\dfrac{s}{(s-a)^3}$	$\left(t + \dfrac{at^2}{2}\right) e^{at}$
20.	$\dfrac{s^2}{(s-a)^3}$	$\left(1 + 2at + \dfrac{a^2t^2}{2}\right) e^{at}$
21.	$\dfrac{1}{(s^2+b^2)^2}$	$\dfrac{\sin bt - bt\cos bt}{2b^3}$
22.	$\dfrac{s}{(s^2+b^2)^2}$	$\dfrac{t\sin bt}{2b}$
23.	1	$\delta(t)$
24.	e^{-cs}	$\delta_c(t)$
25.	$\dfrac{e^{-cs}}{s}$	$h(t-c)$
26.	$\dfrac{1}{s^\alpha}$ $(\alpha > 0)$	$\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$

Table C.2: **Table of Laplace Transforms**

	$F(s)$	$f(t)$
	Laplace Transforms of periodic functions	
27.	$\frac{1}{1-e^{-sp}}\int_0^p e^{-st} f(t)\,dt$	$f(t)$ where $f(t+p)=f(t)$ for all t
28.	$\frac{1}{s(1+e^{-sc})}$	$\mathrm{sw}_c(t)$
29.	$\frac{1}{1-e^{-sp}}\int_0^p e^{-st} t\,dt$	$\langle t\rangle_p$

Table C.3: **Table of Convolutions**

	$f(t)$	$g(t)$	$(f*g)(t)$
1.	1	$g(t)$	$\int_0^t g(\tau)\,d\tau$
2.	t^m	t^n	$\frac{m!n!}{(m+n+1)!}t^{m+n+1}$
3.	t	$\sin at$	$\frac{at-\sin at}{a^2}$
4.	t^2	$\sin at$	$\frac{2}{a^3}(\cos at-(1-\frac{a^2t^2}{2}))$
5.	t	$\cos at$	$\frac{1-\cos at}{a^2}$
6.	t^2	$\cos at$	$\frac{2}{a^3}(at-\sin at)$
7.	t	e^{at}	$\frac{e^{at}-(1+at)}{a^2}$

Table C.3: **Table of Convolutions**

	$f(t)$	$g(t)$	$(f * g)(t)$
8.	t^2	e^{at}	$\frac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$
9.	e^{at}	e^{bt}	$\frac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$
10.	e^{at}	e^{at}	te^{at}
11.	e^{at}	$\sin bt$	$\frac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$
12.	e^{at}	$\cos bt$	$\frac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$
13.	$\sin at$	$\sin bt$	$\frac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$
14.	$\sin at$	$\sin at$	$\frac{1}{2a}(\sin at - at\cos at)$
15.	$\sin at$	$\cos bt$	$\frac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$
16.	$\sin at$	$\cos at$	$\frac{1}{2}t\sin at$
17.	$\cos at$	$\cos bt$	$\frac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$
18.	$\cos at$	$\cos at$	$\frac{1}{2a}(at\cos at + \sin at)$
19.	$f(t)$	$\delta_c(t)$	$f(t-c)h(t-c)$

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